A Simple Harmonic Universe

Shamit Kachru, talk at Cornell meeting, July 2011

Work (soon to appear) with:
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I. Introduction

The Lambda-CDM cosmology, with inflation as a precursor, is supported by an increasingly intimidating amount of data.

In terms of conceptual issues in cosmology, a period of inflation can explain not only the solution of the horizon and flatness problems, but also the density perturbations which generated all of the structure in the Universe we see today.

It is increasingly obvious that inflation + Lambda-CDM is the correct model of cosmology for our Universe.

This talk will not be about that kind of Universe. I will instead focus on two conceptual questions.

i) We all know that the "singularity theorems" of Penrose and Hawking, guarantee that the Universe began with a singularity.

This is not quite true.

Consider, for simplicity, the FLRW cosmologies:

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right) .$$

To prove the singularity theorems, one is required to assume an energy condition. That is, one is required to assume that:

$$T_{\mu\nu}v^\mu v^\nu \geq 0$$

for some class of vectors v. Now, for the Universes with negative or vanishing curvature, k=-1 or 0, one can get by with the "Null energy condition" -- v is just required to be a future-pointing null vector field. This condition is in agreement with everything we know about macroscopic sources our Universe.

But c.f. Creminelli, Luty, Nicolis, Senatore;...

In terms of equations of state for perfect fluids, for instance, this boils down to the condition:

$$p = w\rho$$

$$w \geq -1$$

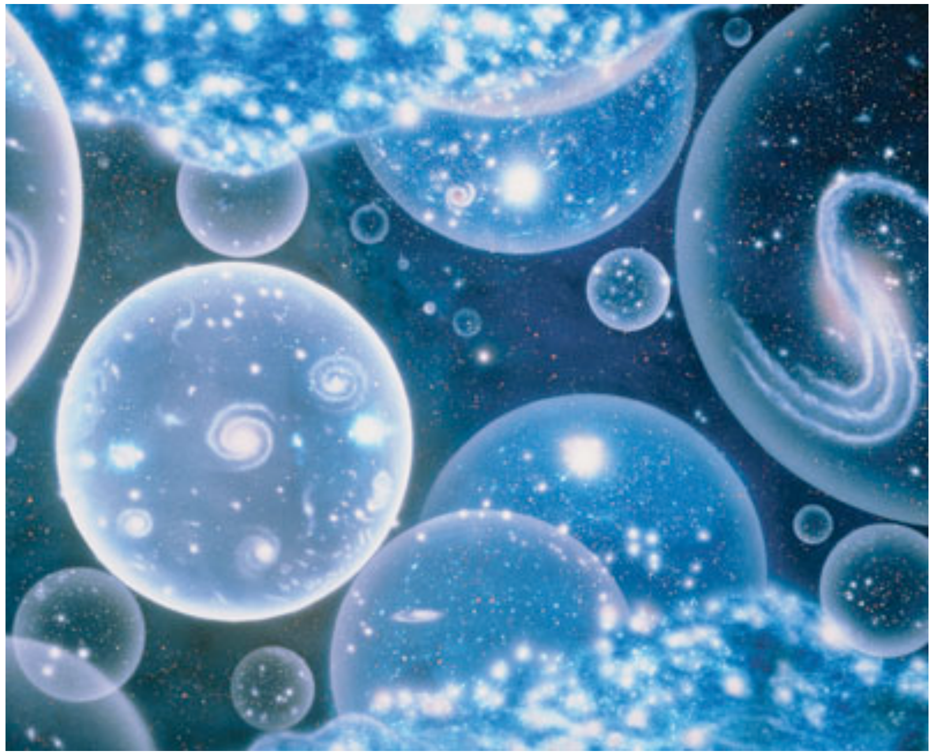
No problem.

For k=+1, things are a bit more confusing. The singularity theorems require one to assume the so-called "strong-energy condition." In terms of w, this is basically requiring that w should satisfy $w \geq -\frac{1}{3}$.

We know, essentially for sure, that the strong energy condition is violated in our Universe, and by many reasonable toy physical models as well.

So, let us ask a theoretical question: Can we make singularity-free, eternal Universes?

The urgency of this kind of question is exacerbated by the following vexing fact. One might have thought that in the string theory landscape, with eternal inflation preceeding our exit into the current (likely metastable) vacuum, there would be no need for an initial singularity:



Theorems of Borde, Guth and Vilenkin guarantee (with reasonable assumptions) that this ain't so.

Inflationary spacetimes are not past-complete

Arvind Borde,[1,2] Alan H. Guth,[1,3] and Alexander Vilenkin[1]

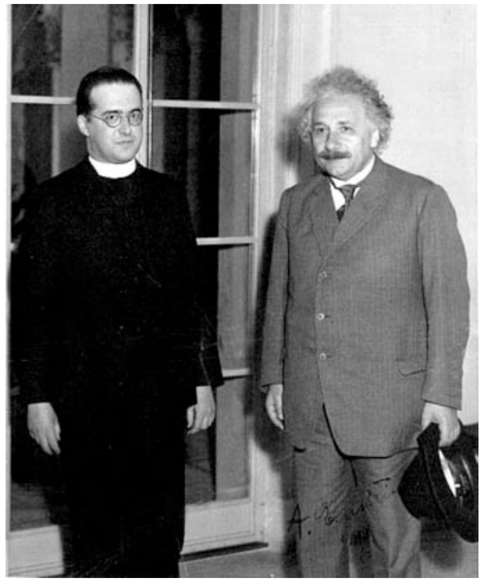
So, what are we to do about the initial singularity?

In this talk, I will make an effort to design reasonable (but not realistic, yet), eternal cosmologies with no singularities. I will fail, but the most unavoidable failure mode is more subtle than that which destroys k=0,-1 cosmologies.

One can then view one goal of this talk, as being a desire to either use k=+1 to evade the singularities, or to motivate a physically useful extension of the theorems to this case.

ii) There has been much interest in the question of whether the Universe can undergo one (or more) instances of a big crunch, without ending the cosmological evolution.

This started, as far as I know, with the work on the "Phoenix universe" (1931) by Lemaitre (see also very interesting papers by Tolman).



Of his earlier work, on the correct FLRW cosmologies which describe our expanding Universe to good approximation, Einstein said:

"Your math is correct, but your physics is abominable."

My hope is that at least the mathematics in this talk will be correct.

One important criterion I will have for what I mean by a bouncing cosmology: there must not be any place in the cosmological evolution where the equations break down and one is forced to assume some boundary conditions which parametrize unknown high-energy physics. c.f. Ellis, Maartens

Such breakdown occurs in many ambitious scenarios. I will instead make the following working definition: any solution where the Universe contracts from a maximal size >> the minimal size (say, megaparsecs across compared to GUT scale size), then re-expands, can be said to "bounce."

This is reasonable because it would appear to "bounce" to any macroscopic observer; and yet since the minimal size can be GUT length >> Planck length, we can remain in the regime of validity of general relativity.

II. General analysis of FLRW equations

So, let's start at the beginning. The Einstein equations applied to an FLRW cosmology yield:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \right)$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3} G\rho - \frac{k}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G \left(\rho + 3p\right)$$

Let us assume, for simplicity, that we have a Universe with just cosmological constant and some stress-energy source obeying the equation:

$$p = w\rho \, .$$

It will be important later that we NOT use perfect fluid sources....

Then if the parameter w is a constant during cosmological evolution, we'll find (with some c>0):

$$\rho_m = c\, a^{-3(1+w)}$$

just from the equations of energy-momentum conservation.

So our system of equations becomes:

$$
\begin{aligned}
\frac{\ddot{a}}{a} &= -\frac{4\pi}{3} G_N (1+3w)\rho + \frac{\Lambda}{3} \\
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3} G_N \rho - \frac{K}{a^2} + \frac{\Lambda}{3} \\
\rho &= \frac{c}{a^{3+3w}}
\end{aligned}
$$

Question: do these equations admit solutions that oscillate between a maximal and minimal scale factor? If so, one must solve:

$$\dot{a}^2 = -K + \frac{8\pi}{3}\frac{G_N c}{a^{3w+1}} + \frac{\Lambda}{3}a^2 = 0$$

at the maximal and minimal value of a(t).

Solution: $K = +1 \ , \ \Lambda < 0 \ , \ -1 < w < -\frac{1}{3} \, .$

Thus it seems we can evade the inflationary singularity theorems using ordinary and well-understood sources!

- a_- produced by K and ρ, while a_+ produced by Λ and ρ
- we require

$$\frac{|\Lambda|}{3} \ll \left(\frac{8\pi}{3} G_N \, c\right)^{2/(3|w|-1)} \ll M_{Pl}^2 \quad \Rightarrow \quad M_{Pl}^{-1} \ll a_- \ll a_+$$

Under these conditions, it is automatic that the second derivative of a(t) has the right behavior at the maximum and minimum to produce the desired oscillations.

III. A Simple Harmonic Universe

While the qualitative features of the solutions are not dis-similar for all w in the allowed range for bouncing cosmologies, for w=-2/3 (c.f. networks of domain walls) the solutions are particularly simple.

$$\ddot{a} + \frac{|\Lambda|}{3} a = \frac{4\pi}{3} G_N c$$

$$\Rightarrow \quad a(t) \quad = \quad \frac{1}{\sqrt{\gamma}\omega}\left(1 + \sqrt{1-\gamma}\cos(\omega t)\right)$$

$$\omega \quad \equiv \quad \sqrt{\frac{|\Lambda|}{3}} \, , \; \gamma \equiv \frac{3|\Lambda|}{(4\pi G_N c)^2}$$

Here, the parameter γ plays an important role:

$$\gamma \sim \frac{a_-}{a_+}$$

In the limit where this parameter is of order unity, we will find one class of behaviors; while for dramatic bounces, we'll find another.

Finally, as conditions on our parameters, we should impose:

$$\frac{4\pi G_N}{|\Lambda|} c \pm a_0 \gg 1\,.$$

$$G_N |\Lambda| \ll 1\,.$$

Under these conditions, semi-classical general relativity is a good approximation. Very naively, these Universes cycle through "crunches" and "bangs" forever.

For more general values -1 < w < -1/3, it is harder to give a useful closed-form solution of the equations. The second order FLRW equation of motion becomes:

$$\ddot{a} + \frac{|\Lambda|}{3}\left(a + \frac{4\pi G_N c}{|\Lambda|}\,\frac{1+3w}{a^{2+3w}}\right) = 0\,.$$

with implicit solution:

$$\int_1^{a(t)} \frac{du}{c_1 - \frac{|\Lambda|}{3}u^2 + \frac{8\pi}{3}G_N c\,u^{-3w-1}} = \pm|t + c_2|$$

Simple numerics shows that there are qualitatively similar oscillating solutions over the whole range up to w=-1/3, where they become necessarily singular.

What should we worry about next?

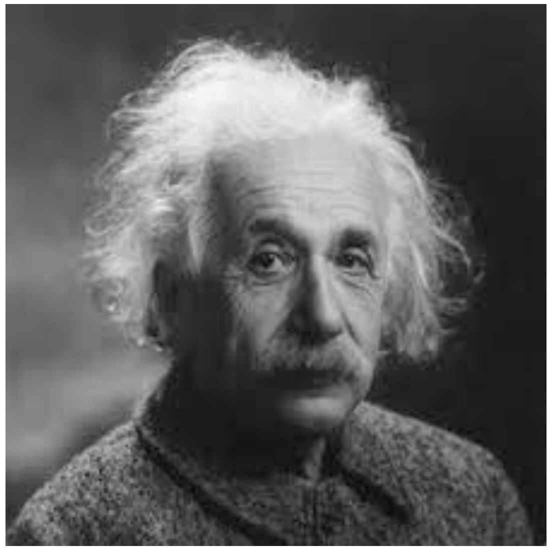
IV. Stability

Two obvious classes of potential instabilities to discuss:

1) Homogeneous perturbations

Recall instability of Einstein's static universe:

- The dust energy density and size of the universe have to be tuned to match the c.c.
- Any deviation causes a large instability.



Our situation will be better than this, but is more nuanced. Clearly, generic small perturbations of c.c., w, or the amount of matter present, do not destabilize us.

The most general homogeneous perturbation of the scale factor takes the form:

$$ds^2 = -dt^2 + \sum_{i=1}^{3} a_i^2(t)\, \sigma_i^2$$

where these are the "Maurer-Cartan" forms on the three-sphere:

$$\begin{aligned}
\sigma_1 &= -\sin\psi\, d\theta + \cos\psi\ \sin\theta\, d\phi \\
\sigma_2 &= \cos\psi\, d\theta + \sin\psi\ \sin\theta\, d\phi \\
\sigma_3 &= d\psi + \cos\theta\, d\phi\,.
\end{aligned}$$

$$0 \le \psi \le 4\pi,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi.$$

We find that the wave equation governing the anisotropic modes is the same as that for a scalar with momentum l=2 on the sphere. We discuss that in detail shortly.

2) Inhomogeneous perturbations

We might also expect possible inhomogeneous instabilities. These are not always disastrous (c.f. collapse of dark matter into halos in our Universe, leaving large-scale homogeneous and isotropic). And free-streaming and/or chaotic mixing can help with them. But they must be analyzed.

(References: Bruni, Dunsby, Ellis; Mukhanov et al.; ...)

One should worry about scalar, vector and tensor perturbations of the metric and background:

$$ds^2 = -\left(1 + 2\Phi(t,x)\right) dt^2 + a^2(t)\left(1 - 2\Psi(t,x)\right) \tilde{g}_{ij} dx^i dx^j \,,$$

For the case of e.g. perfect fluids, we would find:

$$\Phi = \Psi \quad ,$$

$$\delta R_{\mu\nu} = -8\pi G \delta S_{\mu\nu}$$

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}_{\ \lambda} \ , \quad \delta S_{\mu\nu} = \delta T_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \delta T^{\lambda}_{\ \lambda} - \frac{1}{2} \delta g_{\mu\nu} \bar{T}^{\lambda}_{\ \lambda} \, .$$

with the energy-momentum perturbations given explicitly by:

$$\begin{aligned} \delta T_{00} &= 2\, \bar{\rho}\, \Psi + \delta\rho \\ \delta T_{i0} &= -(\bar{\rho} + \bar{p}) \delta u_i \\ \delta T_{ij} &= -2a^2(t) \tilde{g}_{ij}\, \bar{p}\, \Psi + a^2(t) \tilde{g}_{ij}\, \delta p \, . \end{aligned}$$

The end result is:

$$\ddot{\Psi} + 4\ \frac{\dot{a}}{a}\dot{\Psi} + \left(2\,\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{K}{a^2}\right)\Psi = 4\pi G\,\delta p$$

$$\frac{1}{a^2}\tilde{\nabla}^2\Psi - 3\frac{\dot{a}}{a}\dot{\Psi} - 3\left(\frac{\dot{a}^2}{a^2} - \frac{K}{a^2}\right)\Psi = 4\pi G\,\delta\rho$$

$$\partial_i\left(\dot{\Psi} + \frac{\dot{a}}{a}\Psi\right) = 4\pi G(\bar{\rho} + \bar{p})\delta u_i\,.$$

For e.g. a perfect fluid, one would have:

$$\delta p = c_s^2\delta\rho\ ,\ c_s^2 = \frac{\partial p}{\partial\rho}\Big|_{\rho=\bar{\rho}}\,.$$

Then one can simplify the system of equations to:

$$\ddot{\Psi} + \frac{\dot{a}}{a}(4 + 3c_s^2)\dot{\Psi} + \left(2\,\frac{\ddot{a}}{a} + (1 + 3c_s^2)\left(\frac{\dot{a}^2}{a^2} - \frac{K}{a^2}\right)\right)\Psi - \frac{c_s^2}{a^2}\tilde{\nabla}^2\Psi = 0\ \ .$$

An important point is the following: If we really had a perfect fluid with $c_s^2 < -1/3$, there would be catastrophic short distance instabilities.

Intuitively, this is because there would be negative pressure, leading to instabilities which are worse and worse on shorter and shorter distance scales. This is visible from the sign of the Laplacian term in the final differential equation.

Now, we certainly need a stress-energy source which enters in the Friedmann equations and has w < -1/3 to support our simple harmonic Universe. Are we in trouble?

We are rescued by a simple, physical observation, discussed at length in this paper:

ITP-SB-97-22

Is the Dark Matter a Solid?

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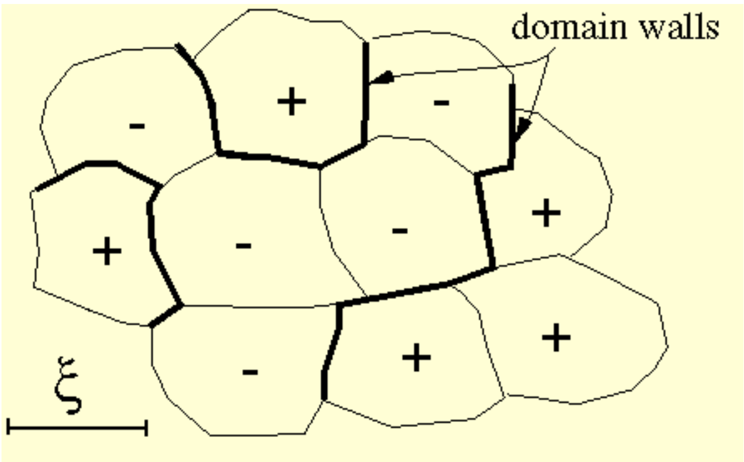
Their simple observation, visible in their abstract, is that:

For a perfect fluid negative presure leads to instabilities that are most severe on the shortest scales. However, if instead the dark matter is a solid, with an elastic resistance to pure shear deformations, an equation of state with negative presure can avoid these short wavelength instabilities. Such a solid may arise as the result of different kinds of microphysics. Two possible candidates for a solid dark matter component are a frustrated network of non-Abelian cosmic strings or a frustrated network of domain walls. If these networks settle down to an equilibrium configuration that gets carried

So for instance, a frustrated network of domain walls contributes in the Friedmann equations with w=-2/3, but has a large positive sound speed and no dangerous instabilities to inhomogeneous perturbations.



For detailed investigations, see:
Battye, Moss;
Battye, Pearson;
Battye, Pearson, Moss

This is precisely the case where our simplest (sinusoidal) oscillatory solution obtains.

So it seems we have dodged part of bullet 1), and any dangers arising in the high momentum regime of 2). However, now we need to be more careful.

V. Stability in models of O(1) vs tiny γ

Let's consider for simplicity a scalar field (or the simplest graviton modes) in the periodic solution in conformal time. This is relevant to the mixmaster modes we discussed earlier, for instance.

$$ds^2 = a(\eta)^2(-dt^2 + \tilde{g}_{ij}(x)dx^i dx^j)$$

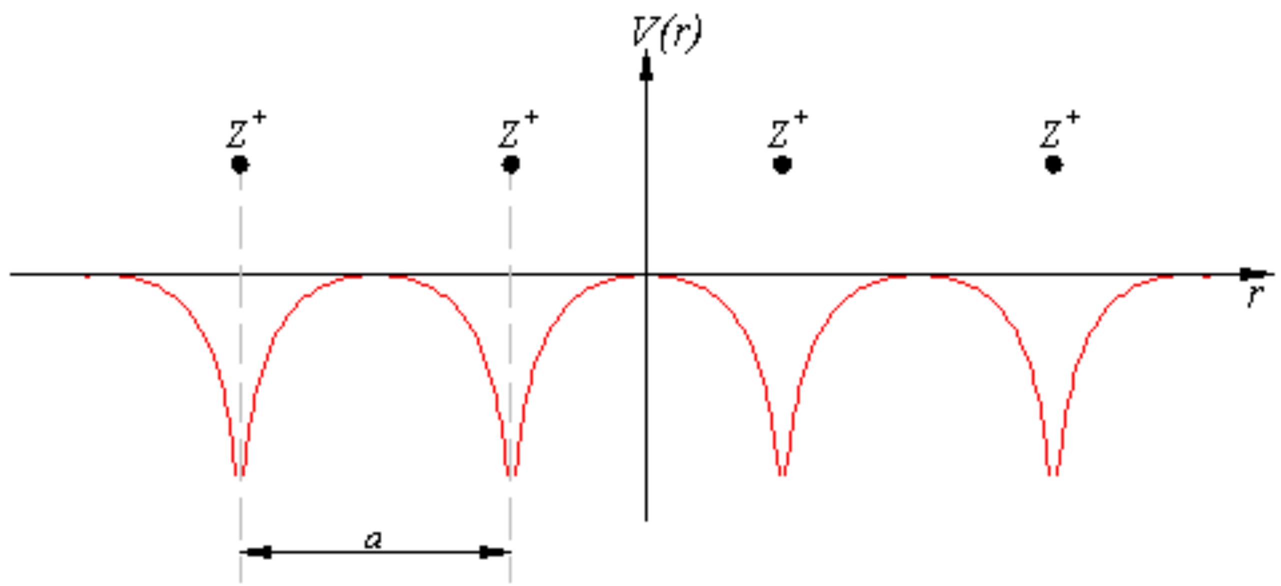
$$a(\eta) = \frac{1}{\omega}\frac{\sqrt{\gamma}}{1-\sqrt{1-\gamma}\cos(\eta)} \; .$$

The action and EOM are given by:

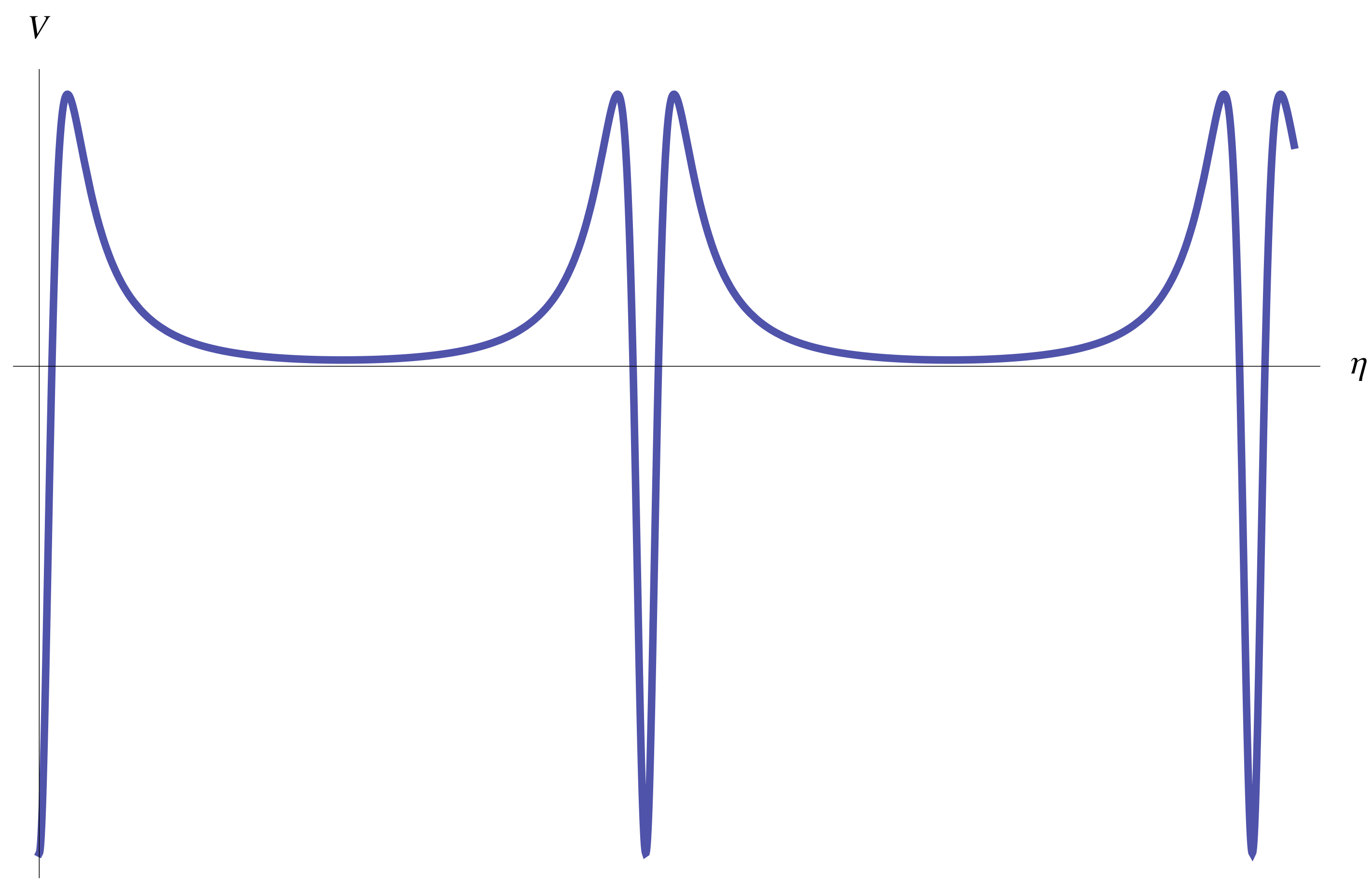
$$S = \int \; d^4x \; a(\eta)^2 \left((\partial_\eta \phi)^2 - (\partial_i \phi)^2\right)$$

$$\phi'' + 2\frac{a'}{a}\phi' + l(l+2)\phi \;=\; 0$$

Amusingly, by a change of variables, given the periodicity of the scale factor, this problem can be mapped to the quantum Schrodinger problem of an electron moving in 1d in a periodic potential, analyzed by Bloch many years ago.



The actual potential that arises in the case with sinusoidal scale factor is not quite of the "Dirac comb" form familiar from simple treatments of Bloch waves and conduction bands. Instead one finds:



There are then three relevant parameter ranges in which to analyze the differential equation

$$\phi'' + 2\frac{a'}{a}\phi' + l(l+2)\phi \; = \; 0$$

depending on the size of l relative to γ :

$\surd$ Instead of periodic b.c., we want to give initial conditions for χ

$\surd$ Set of decoupled harmonic oscillators in each cycle

$\surd$ Patch sols across 'barriers' and 'wells' of the potential

► The homogeneous mode exhibits linear growth,

$$\phi(\eta) \sim \frac{\eta}{\gamma^2}$$

► For intermediate momenta $k < 1/\sqrt{\gamma}$,

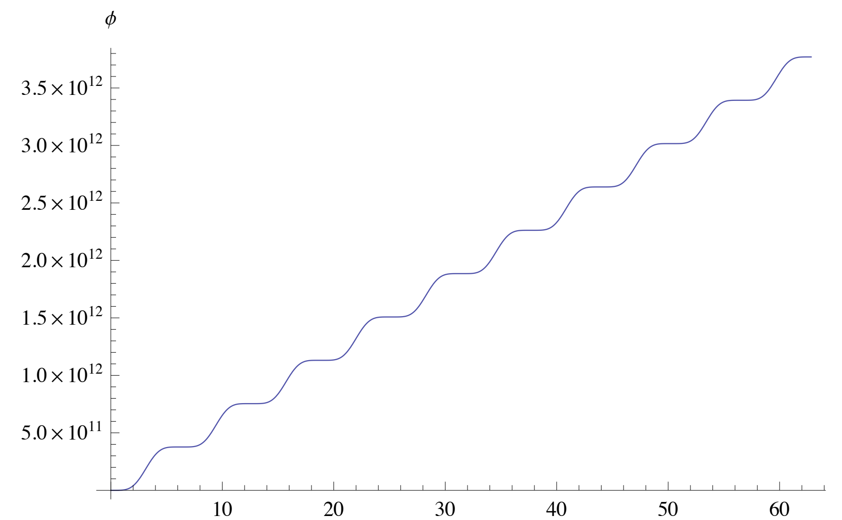
$$\phi(\eta) \sim \exp\left[\sqrt{1 - \frac{k^2}{k_c^2}}\ \eta\right] \quad , \quad k_c^2 \sim 1/\gamma$$

► At high momenta, modulated Minkowski modes,

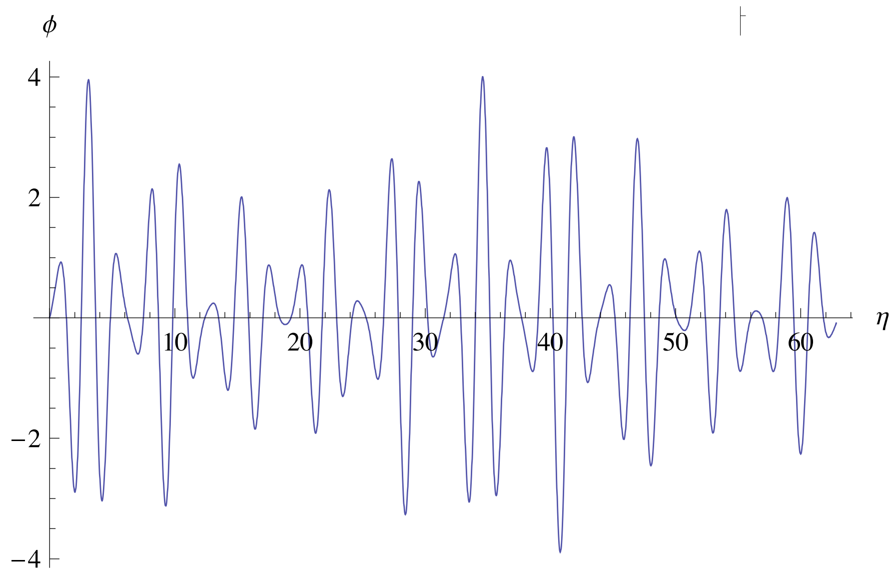
$$\phi(\eta) \sim (\sin \eta)\ e^{ik\eta}$$

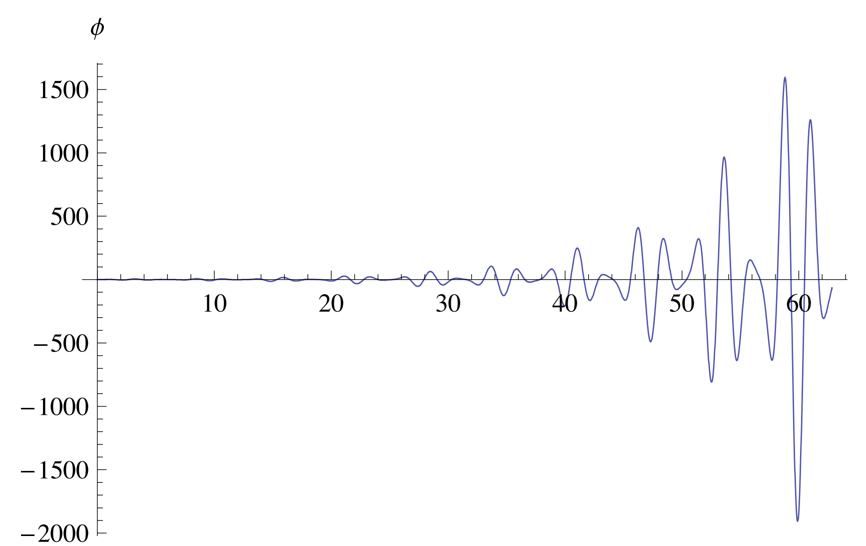
Instantiating plots:

Linear growth of homogeneous mode



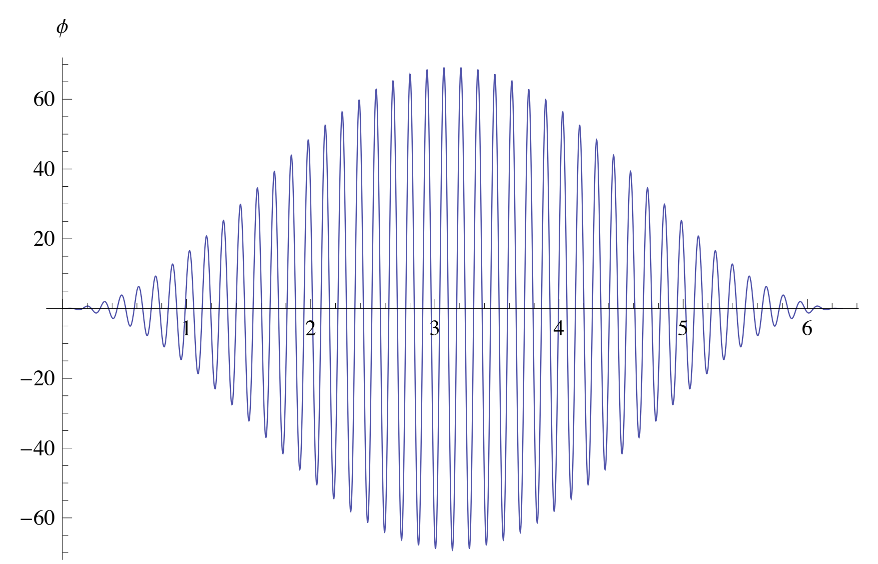
$\gamma \sim 1$





$\gamma \ll 1$

Oscillatory high-momentum modes



As a check on the numerics, we can exactly solve some of the cases/limits. E.g. the exact homogeneous mode is:

$$\phi(\eta) = \phi(0) + \phi'(0)$$
$$\times \quad \frac{(3-\gamma)\eta - 4\sqrt{1-\gamma}\,\sin(\eta) + \frac{1}{2}(1-\gamma)\sin(2\eta)}{2(1-\sqrt{1-\gamma})^2}$$

which exhibits the linear growth with conformal time, and matches the numerical solution very well.

Some physics points:

* The linear growing homogeneous mode is not an instability. It reflects the way that two sinusoidal functions of slightly different frequencies move away from one another, in perturbation theory.

* For $\gamma \sim 1$ there are no instabilities, then. The single case with meaningfully growing modes in the plots does not occur. (Don't worry, I'm a pessimist. I will come back to worry about this case.)

* For $\gamma \ll 1$ we have some exponentially growing modes! What is the ensuing physics?

VI. The end of time

For the oscillating Universes with large ratio of maximal to minimal scale factor, we believe the growing modes implement the fate envisioned in the ancient texts:

"Death is as sure for that which is born, as birth is for that which is dead. Therefore grieve not for what is inevitable." - Bhagavad-Gita

A. Classical death

The rate of exponential growth for the growing modes:

$$\phi_l(N) \sim \phi_0 \; \exp\left(\sqrt{1-\frac{l^2}{l_c^2}} \times N\right)$$

after N oscillation cycles, will cause fatal difficulties for our Universe (or at least, for our approximations).

The pertinent physical question is: at what point does the energy in these modes compete at O(1) with the background we started with?

The ratio of the energy carried by the scalar field to e.g. the energy in the cosmological term, is given by:

$$\sum_l \frac{a^2 l(l+2)\phi^2}{M_P^2 a^4 |\Lambda|} \sim \frac{\gamma}{M_P^2} \int^{k_c} dl \; l^2 \phi^2 \; .$$

One can evaluate the integral by a saddle point approximation, yielding a dominant momentum regime:

$$l^2_{saddle} = \frac{1}{N} l_c^2$$

So, the energy ratio turns out to be:

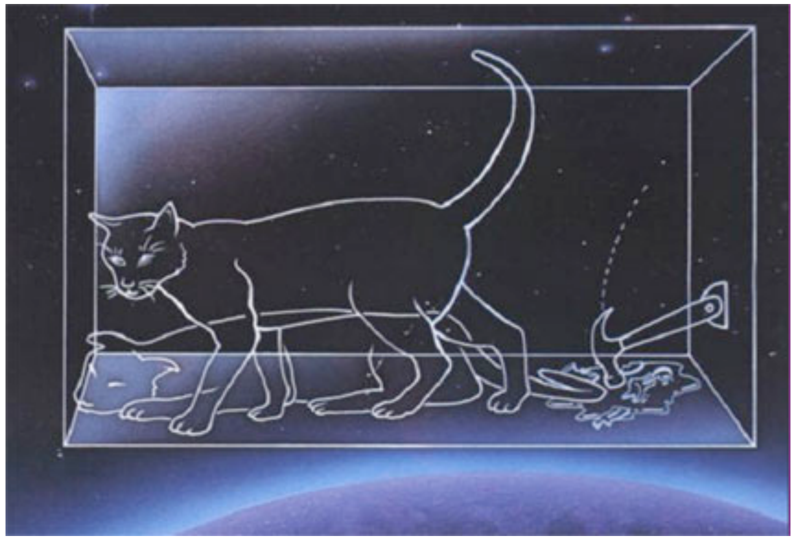
$$\frac{\epsilon_\phi}{|\Lambda|} \sim \gamma l_c^2 \frac{\phi_0^2}{M_P^2} \; \exp\left(N - \log N\right) \; .$$

Hence, backreaction becomes important after a number of cycles given by:

$$N_c \sim 2\,\log\left(\frac{M_P}{\phi_0}\right)\ .$$

Classical tuning can yield many cycles, but at the expense of tuning the initial magnitude of the scalar field.

B. Quantum death



There are various ways one could incorporate quantum mechanics here. The quickest & dirtiest, fitting to an informal talk, is the following.

* The scalar field is just a collection of harmonic oscillators.

* Then in quantum theory, we should impose the canonical commutation relations on the canonically normalized cousin of the scalar:

$$\chi \equiv a(\eta)\phi, \quad [\chi(\theta), \partial_\eta \chi(\theta'] \;=\; i\delta^{(3)}(\theta - \theta')$$

* We are free to choose whichever quantum state we want. Let us suppose we choose the oscillator ground state characteristic of the system at a scale factor a(t). Then a simple computation shows:

$$a^2\phi_0^2 \sim 1 \;.$$

This is not a surprise. Quantum mechanics imparts an RMS expectation value to the position of a harmonic oscillator.

We can give a state-choice dependent quantum bound by plugging now into:

$$N_c \sim 2\,\log\left(\frac{M_P}{\phi_0}\right)\ .$$

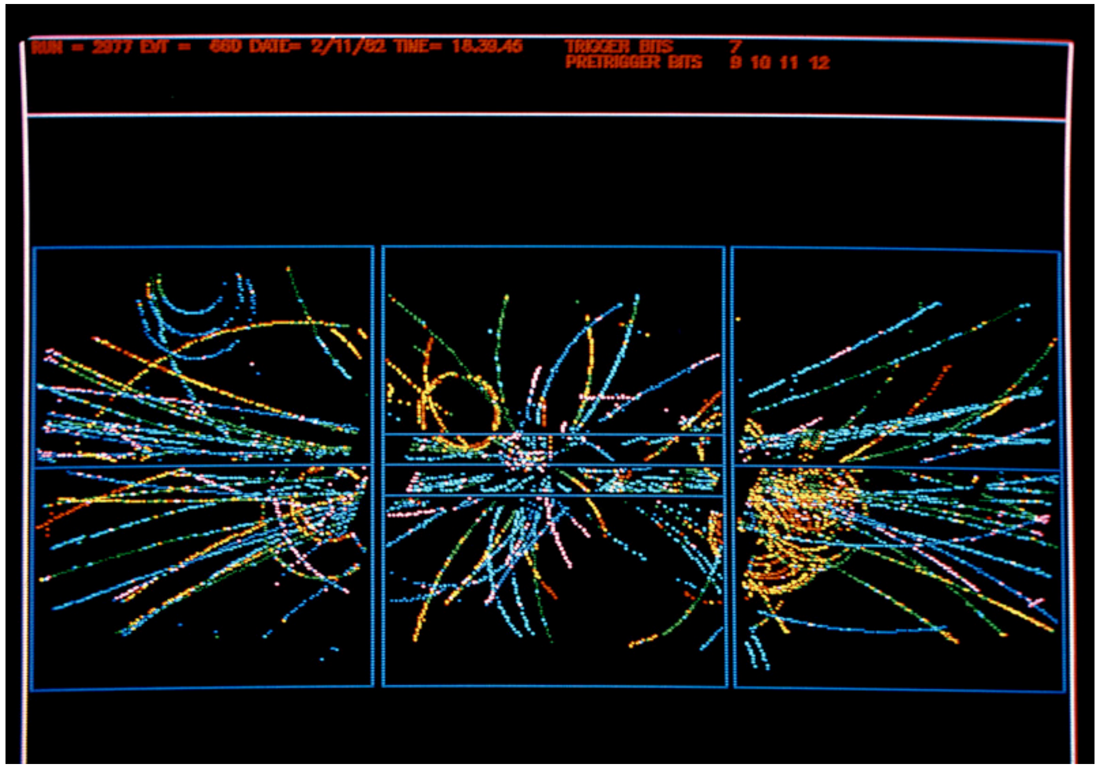
The most extreme bounds (least/most stringent) are obtained by choosing the ground state at maximal/minimal scale factor. Result at maximum:

$$N_c \sim 2\,\log\left(\frac{M_P}{\sqrt{\gamma}\omega}\right)\ .$$

(The result at the opposite extreme differs in a very simple way).

Crucial point: By tuning the cosmological term in Planck units (to be negative but of very small magnitude), one can obtain an arbitrarily large number of cycles.

Interpretation of eventual quantum death: “particle production.”



Any cosmology contains, among other things, gravity and gravitons. If nothing else, graviton production can occur as the Universe expands and contracts. This gas of gravitons has $w=1/3$; when present in sufficiently appreciable number density, these gravitons will cause a crunch when one reaches minimal scale factor, instead of acquiescing to a "bounce."

Note that the way we estimated both classical and quantum death is applicable only to the Universes with "large" ratio of maximal to minimal scale factor. I personally suspect that by combining the covariant entropy bound with more traditional considerations, one can prove a quantum singularity theorem in general for closed Universes.

VII. Conclusion/Questions

1. Can we prove a quantum singularity theorem for closed Universes?

2. In the cases with large ratio of maximal to minimal scale factor, can we embed a realistic cosmology into one cycle?

3. At the level we've worked so far, we do have some apparently stable, eternal solutions. Can we envision one of these in the deep past, tunneling to a realistic inflationary Universe?

4. We ignored entropy production due to our classical source with $w < -1/3$. Would this always dominate the effects we discussed, and lead to earlier problems?

Finally:

5. The relation of the quantum mechanics of particle production in these geometries strongly suggests that one consider special periodic quantum states as preferred. (Note that for us it was natural to impose initial boundary conditions on the field, not periodic BC; but special choices...).

Could there by periodic wavefunctions of the Universe in such geometries that provide natural choices of boundary conditions, and give eternal cosmologies?

c.f. Hartle/Hawking, which is ad hoc but "natural"...