SUBSTITUTE TA, P2214
SECTION 205 (Csaba Csáki)
THURSDAY 10:10 - 11:00, ROCKEFELLER 104

29 APRIL 2010

PLAN: 10:10 - 10:30 QUIZ # 10
10:30 - 11:00 SOLVE PROBLEM #5, maybe #6 ON HW #13

↳ READ PROBLEM
5 MIN TO TALK ABOUT IT COOPERATIVELY
DISCUSS SLOWLY & ~~[illegible]~~ w/ time for q's
@ each step.

1. HAND OUT QUIZ + PROCTOR. 20 MINS.
DON'T FORGET TO PICK UP QUIZZES!!

2. BRIEFLY: BIG PICTURE

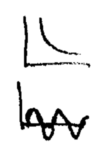
YOU'VE BEEN TAKING YOUR FIRST STEPS IN QUANTUM MECHANICS (WAVE MECHANICS). THIS WEEK: SOLVING THE SCHRÖDINGER EQ.

WANT TO UNDERSTAND: QUANTUM MECHANICAL TUNNELING
ie PARTICLES IN CLASSICALLY FORBIDDEN REGIONS

TECHNIQUES: HOW TO PATCH TOGETHER SOLUTIONS OF THE WAVE EQUATION (ODE + BC)
→ this is important & shows up over & over in physics & engineering

SANITY CHECK

$$\frac{d^2}{dx^2} f(x) = \pm k^2 f(x) \Rightarrow \begin{cases} f \sim e^{kx} \\ f \sim e^{ikx} \end{cases} \rightarrow \cos \text{ \& } \sin$$

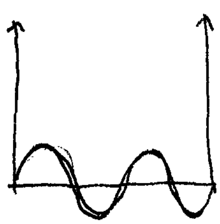


$\boxed{\text{UNDERSTAND CONNECTION BTWN SIGN IN ODE \& OSC VS EXP!}}$

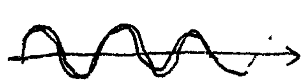
or just say $f \sim e^{kx}$ w/ $k \in \mathbb{C}$

things you know:

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx} + U(x)\psi = E\psi}$$



∞ WELL → QUANTIZATION!
only particular freq!



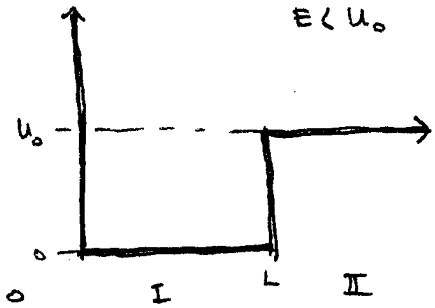
$U_0 = 0$

free particle
⇒ no constraint

[Q: WHAT IF $E < U_0$?
then we go from cos/sin → exp]

what does this mean physically?

HW #13 PROBLEM #5



0) ... What is form of ψ in $x<0$?
a) WHAT IS FORM OF ψ IN $0<x<L$?
b) WHAT IS FORM OF ψ IN $x>0$
c) WHAT ARE BC @ L
→ WHAT ARE ALLOWED ENERGIES?
d) TAKE $U_0 \to \infty$, COMPARE E's TO ∞ SQ. WELL

{ 5 minutes of collaboration }

PRE DISCUSSION: WHAT IS OUR INTUITION?

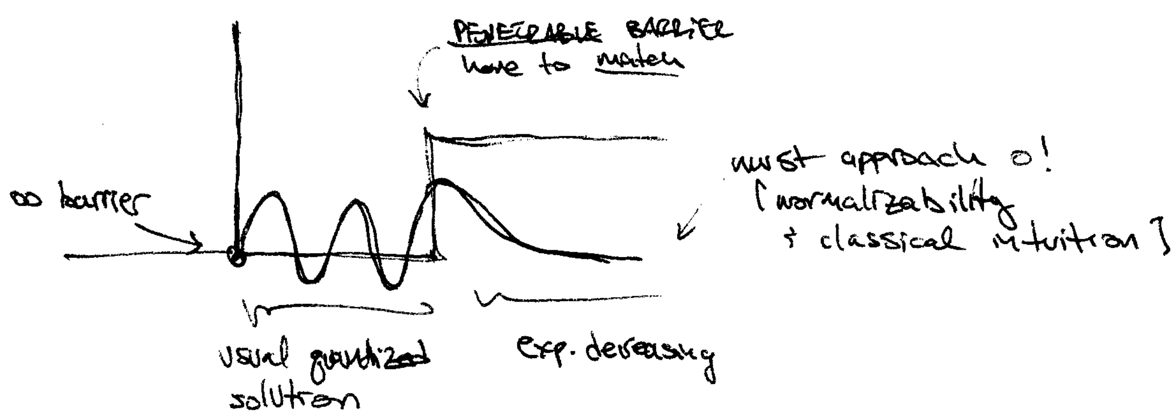
$x<0$: CLASSICALLY FORBIDDEN
QUANTUM MECHANICALLY ALSO FORBIDDEN (req. ∞ E)

→ in fact, this might seem somewhat mysterious.
QM is a whole new ball game, why can't we tunnel?
[can see from Schrödinger eq, but we'll gain more insight as we consider the finite edge]

$0<x<L$: LOOKS LIKE SQUARE WELL.
EXPECT SIMILAR SOLUTION. ("classical")

$x>L$: CLASSICALLY FORBIDDEN
... QUANTUM MECHANICALLY ... CAN LEAK INTO FORBIDDEN REGION!

WHAT DO WE EXPECT ψ to look like?
→ it is very important to have an idea what the solution ought to look like!!
[THIS IS THE PART THAT REQUIRES SOME CREATIVITY, BUT IN REAL QUESTIONS IT'S ALWAYS THE CRITICAL STEP]



IDEA: SOLVE IN EACH REGION & PASTE TOGETHER.
have to solve for 4 coef. → need 4 BC!

a) $\underline{0<x<L}$: $U(x)=0$

$\underline{BC}$: $\psi(0)=0$ [does everyone understand why?]

SCHRÖDINGER IN THIS REGION:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad \Rightarrow \quad \psi'' = -\frac{2m}{\hbar^2}E\,\psi$$

expect $\underline{\text{sinusoidal}}$ solutions [WE OF COURSE KNOW THIS FROM $\underline{\text{PHYSICAL}}$ INTUITION]

$$\psi(x) = A_1 e^{ikx} + A_2 e^{-ikx} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

[ONLY IN THIS REGION]

$$= (A_1+A_2)\cos kx + i(A_1-A_2)\sin kx$$

as defined in the problem. ✓ (good! we're on the right track!)

$\underline{BC}$: $\psi(0) = 0 = A_1 + A_2$

$$\Rightarrow \boxed{\psi(x) = C\sin kx}$$

call this ψ_I

$C = i(A_1 - A_2)$ is some constant to solve for.

$\underline{\text{SANITY CHECK}}$: STARTED w/ SOL OF WAVE EQ → 2 UNKNOWNS
HAD $\underline{\text{ONE}}$ BC, REDUCE SOL. TO 1 UNKNOWN.

what about other boundary? we'll get to that.

{ ANY QUESTIONS? }

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b) $\underline{x>L}$ $U(x) = U_0$

$$\underbrace{K}_{\text{KAPPA}} = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$$

NOW SCHRÖDINGER EQ. $\underline{\text{CHANGES}}$!

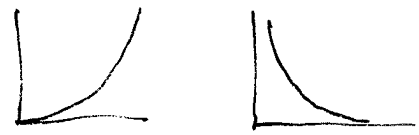
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = \underbrace{(E-U_0)}_{\text{negative}}\psi \quad \Rightarrow \quad \psi'' = \underbrace{\frac{2m}{\hbar^2}(U_0-E)}_{\text{positive}}\psi$$

→ $\underline{\text{exponentials}}$ not sines.

Q. WHAT IS THE BC?

Q. $\underline{\text{WHY}}$? [normalizability — otherwise probability → ∞]

b, contd./ $\psi_{II} = A_3 e^{Kx} + A_4 e^{-Kx}$



note: NEITHER of THESE ARE A SENSIBLE SOLUTION FOR ALL VALUES of x.
→ not normalizable over $\int_{-\infty}^{\infty}$

FORTUNATELY: e^{Kx} is well behaved near $x=0$
e^{-Kx} —— " —— near $x=\pm\infty$

AND: in this case we're in region II: $x>L$
so we don't have to worry about the "divergence" @ $x=0$

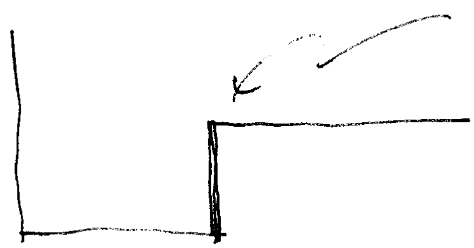
$\int_{II} e^{-Kx}$ is PERFECTLY FINE & NORMALIZABLE

this is the entire content of the "BOUNDARY CONDITION" AT $x\to\infty$: $\psi(\infty) = 0$.

$$\Rightarrow \boxed{\psi_{II} = A_4 e^{-Kx}}$$

Sanity check: AGAIN: sol of sch eq: 2 unknowns
BC (just one): → 1 UNKNOWN.

SO NOW WE HAVE:



have to determine 2 unknowns (A_4 & C) @ this boundary!

also want to determine ENERGY LEVELS ... ie have to combine k & K in some meaningful way.

$\psi_I = C\sin(kx)$ $\psi_{II} = A_4 e^{-Kx}$

c) IMPOSE BC @ $x=L$

AS EXPECTED, TWO BC: $\begin{cases} \psi_I(L) = \psi_{II}(L) \\ \psi_I'(L) = \psi_{II}'(L) \end{cases}$

Q: WHY DO WE NEED THESE?
OTHERWISE ψ'' IS DISCONTINUOUS → SCHRÖ PREDICTS ∞ ENERGY.
doesn't make sense

$$\boxed{\begin{aligned} C\sin(kL) &= A_4 e^{-KL} \\ Ck\cos(kL) &= -A_4 K e^{-KL} \end{aligned}}$$

↓

dividing: $k\dfrac{\cos(kL)}{\sin(kL)} = -K$

$$\Rightarrow \boxed{K = -k\cot(kL)}$$

This is kind of a pain in the ass to solve explicitly!

PARENTHETICAL REMARK (NON-EXAMINABLE!)
HERE'S AN IDEA OF HOW THIS CAN BE SOLVED GRAPHICALLY

note: $K^2 + k^2 = \frac{1}{\hbar^2} 2mU_0 \equiv \beta^2 \quad (*)$

JUST SOME # (CONSTANT)

FIRST: DEFINE: $x = kL$ SO THAT

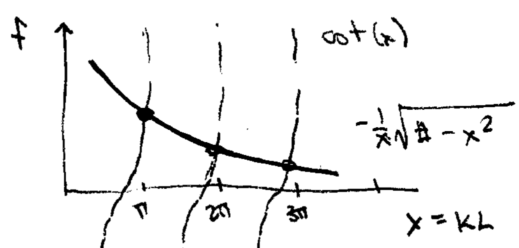
$$\boxed{-\frac{K}{k} = \cot(x)}$$

NEXT: USE $(*)$ TO WRITE $-K/k$ IN A DIFFERENT WAY

$$K^2 = \beta^2 - k^2 \Rightarrow -\frac{K}{k} = -\sqrt{\frac{\beta^2 - k^2}{k^2}} = -\sqrt{\frac{\beta^2 L^2}{x^2} - 1} = \boxed{-\frac{1}{x}\sqrt{\# - x^2}}$$

still some const.

SOLUTION IS CONSISTENT WHEN BOTH BOXES AGREE, SO PLOT:



INTERSECTIONS GIVE ALLOWED VALUES OF x
⇒ ALLOWED VALUES OF k & K
⇒ ALLOWED VALUES OF E

d) TAKE LIMIT $U_0 \to \infty$ & SHOW THAT WE GET ∞ WELL ENERGIES

Q. IS IT CLEAR WHY THIS SHOULD BE TRUE?

$$\mathcal{K} = \underbrace{-k \cot kL}_{\text{INDEP of } U_0}$$

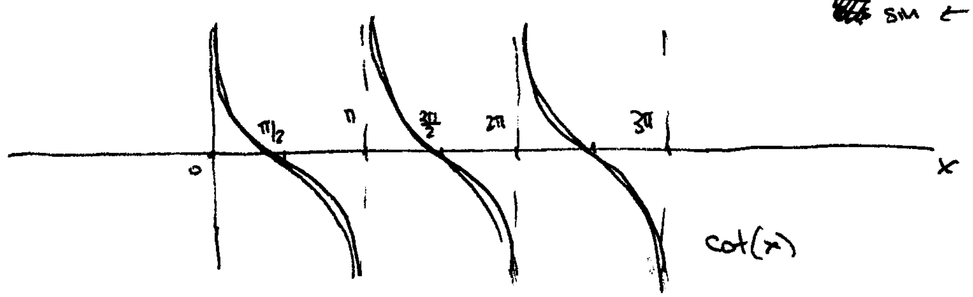
$$\mathcal{K} = \sqrt{\frac{2m(U_0 - E)}{\hbar}}$$

so: $U_0 \to \infty \Rightarrow \mathcal{K} \to \infty$

WE ASSUME k remains FINITE (it MUST)

$\Rightarrow$ for what values of $x = kL$ does $\cot(x) \to -\infty$?

$\cot = \frac{\cos}{\sin}$ ← zeros of sin



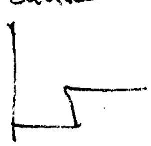
$$\Rightarrow \boxed{kL = n\pi}$$

cf eq. (40.7) for ∞ sq. well.

OTHER THINGS TO MENTION

- WHAT ABOUT THE LIMIT $L \to \infty$?
  WHAT IS YOUR INTUITION?
  CAN YOU SEE HOW WE GET IT? (eg from PARENTHETICAL REMARK)

- ODD WAVEFUNCTION TRICK. IF WE KNEW SOLUTION OF SQUARE WELL, WE COULD HAVE GOTTEN SOLUTION OF



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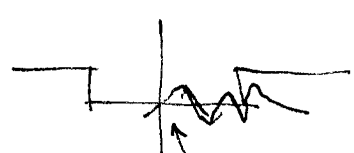

- KOVARS/HALL H.O.
  ↳ cf. § 42.2 OF TEXT.