Anomaly Wars: Return of the Quanta

Joshua Berger

We give a review of anomalous $B + L$ violation in the Standard Model. We discuss both thermal and quantum processes that violate $B + L$. Such transitions are inherently non-perturbative: they involve extended field configurations with non-trivial global topology. We give an overview the two main tools we have for studying these transitions, sphalerons and instatons, and present results obtained using these methods in both a simplified toy model and in an $SU(2)$ gauge theory. We will also discuss the implications of these results for the Standard Model.

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I. INTRODUCTION: SOMETIMES PT JUST DOENS'T CUT IT

The renormalizable SM has a $U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau$ symmetry classically. The $U(1)_{B+L}$ subgroup of this symmetry is, however, anomalous and should be violated by quantum effects. In particular, there should be allowed processes that violate $B+L$. These processes cannot arise at any level in perturbation theory, since a global change in $B+L$ number requires a field configuration with non-trivial topological charge. Any symmetry violating effects must therefore arise due to non-perturbative effects.

One might suspect that such non-perturbative effects would be completely inaccessible computationally. The trick to accessing the necessary non-perturbative computations is to expand about a field configuration that is not the vacuum. The non-perturbative effects that violate $B+L$ can be calculated as perturbative expansions about such configurations. We have already discussed one such possible extended field configuation when we discussed monopoles/solitons. Two different types of extended field configurations are relevant to the study of $B+L$ violation: instantons and sphalerons. Both of these extended field configurations have their conceptual pitfalls.

Instantons are used for calculating non-perturbative tunneling events between different vacua of the theory. Typically, interaction events are centered around a single event in spacetime. When discussing the local interactions of perturbative field theory, the interactions occur *exactly* at single event. An instanton is an extended field configuration that represents a "fuzzed-out" interaction event. The vacuum state in the distant past and distant future are different, with the transition centered around a single spacetime point. We can then perturbatively expand about this instanton configuation. We will see, however, that we can only make a semi-classical approximation using instantons. We only pick out the lowest energy configuration contributing to the transition and then perturb about it. In determining these configurations, we will find a deep connection to topology. We will discuss this point further in Section 3.

Sphalerons are less intuitive. As we will see, instantons are exponentially suppressed for perturbative gauge theories. For the gauge theory of interest, $SU(2)_L$, this exponential suppression is $e^{-16\pi^2/g^2} \sim 10^{-173}$ at zero temperature, which is negligible. For phenomenological purposes, $B+L$ violation occurs more readily at finite temperature. We will relate thermal fluctuations between different $B+L$ states to *unstable* static solutions to the equations of motion that interpolate between field configurations of different effective $B+L$ charge. These unstable configurations are called sphalerons.

The goal of this review talk is to give an overview of the formalism required to understand the non-perturbative physics behind $B+L$ violation. We begin by giving an overview of the vacuum structure of gauge thoery and discuss the relation to topology. We attempt to avoid, whenever possible, rigorous math and take a minimalist approach toward this. The vocabulary will be presented, but we make no formal definitions nor do we prove any results. We continue by discussing the solutions of interest for $B+L$ violation. We then address the questions of thermal and quantum $B+L$ violation in turn. In dealing with thermal fluctuations, we will first a very simple toy example. We will then discuss a more realistic example and mention some subleties. The quantum transitions will be exponentially suppressed by $e^{-16\pi^2/g^2} \sim 10^{-173}$ and will be negligible. Time permitting, we will review this quantum calculation as well. We will conclude by very briefly discussing some implications of these results.

II. ANOMALIES AND WINDING NUMBER: A LOVE STORY

The baryon-number violating instanton and sphaleron are both related to the structure of $SU(2)$ solutions. Generically, in a model with an $SU(2)$ symmetry, which may or may not be spontaneously broken, there exist topologically stable solutions to the equations of motion. The important point to note at this time is that such solutions can carry effective charge under anomalous global symmetries of the theory. The goal of this section is to understand the connection between anomalous symmetries and topologically stable solutions to the classical equations of motion.

We begin by very briefly reviewing anomalies. For a more detailed review, see any good Quantum Field Theory book. We focus on so-called chiral anomalies, which for our purposes means anomalies of $U(1)$ global symmetries. Consider a theory whose classical Lagrangian has a global $U(1)$ symmetry with conserved current j^μ. Suppose further there are fermions ψ_i with charges Q_i under the $U(1)$. The symmetry is anomalous if $\partial_\mu j^\mu \neq 0$ at the quantum level. This statement means that the symmetry is anomalous if there is ever a non-zero matrix element $\langle f|\partial_\mu j^\mu|i\rangle \neq 0$. Assume now that the theory has an gauge symmetry G under which the ψ_i transform in representation r_i. We can then calculate the matrix element $\langle p,\nu,a;k,\lambda,b|\partial_\mu j^\mu|0\rangle$, where a,b label adjoint gauge indices, μ,ν,λ are Lorentz indices, and p,k are the momenta of the outgoing gauge bosons. This matrix element vanishes if the symmetry is not anomalous. A careful calculation reveals that

$$\langle p,\nu,a;k,\lambda,b|\partial_\mu j^{\mu c}|0\rangle = -\frac{g^2}{16\pi^2}\sum_i Q_i C(r_i)\langle p,\nu,a;k,\lambda,b|F^{\mu\nu c}\tilde{F}_{\mu\nu c}|0\rangle, \tag{1}$$

where the generators t_i are in the representation r_i and $\tilde{F}$ is the dual field strength. If the sum of $Q_iC(r_i)$ is non-zero, then the symmetry is anomalous. We have glossed over issues both glaring and subtle, such as understanding how quantum mechanics can violate a symmetry in the first place. These issues are beyond the scope of this talk.

Consider the case of SM baryon and lepton number, with the gauge symmetry being $SU(2)_L$. All the $SU(2)_L$ singlets do not contribute to the possible anomaly. That leaves only $Q_L^{i\alpha}$ for baryon and L_L^i for lepton number, where i is the flavor index and α is the color index. For $SU(2)$, $C(\mathbf{2}) = 1/2$. For the quark doublets, $B_i = 1/3$. For the lepton doublets, $L_i = 1$. We then see that both $U(1)_B$ and $U(1)_L$ are seperately anomalous:

$$\partial_\mu j_B^\mu = -\frac{g^2}{32\pi^2} n_F F\tilde{F}, \qquad \partial_\mu j_L^\mu = -\frac{g^2}{32\pi^2} n_F F\tilde{F}, \tag{2}$$

where n_F is the number of flavors. Notice that the current for $B-L$, $j_{B-L} = j_B - j_L$ is anomaly free. Furthermore, the currents for the relative abundance of each lepton family $L_i - L_j$ is anomaly free. The current for j_{B+L}, however, has an anomaly:

$$\partial_\mu j_{B+L}^\mu = -\frac{g^2}{16\pi^2} n_F F\tilde{F}. \tag{3}$$

Notice that his anomaly depends on the number of flavors n_F that contribute in the triangle diagram loop. If we consider all of the standard model fermions to be massless, this means that the amount of $B+L$ violation is related to the number of flavors of particles in the model. This makes sense, since the relative abundance of each family of lepton number must be preserved: if we were to add an electron to the universe, we would also have to add a muon and a tau. All the left handed fermion doublets in the standard model must be involved in the anomaly. We will see this condition come up in some other ways later.

The quantity $F\tilde{F}$ appears in another context. Consider Yang-Mills with a simple gauge group G which we will take to be $SU(N)$ eventually. This theory has an action (with gauge indices suppressed):

$$S = \int d^4x - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{4}$$

Clearly, if the action is to be finite, $F^{\mu\nu}$ must vanish as $|x| \to \infty$. Equivalently, A^μ must approach a pure gauge solution. But A^μ need not approach the same pure gauge solution in every spacetime direction away from the event. Furthermore, the gauge at spacetime infinity can have non-trivial topology and can affect the structure of the solution near the event. The set of all solutions that are pure gauge is the same as the set of all gauge transformations, which is the set associated with the gauge group G. Thus we are looking for maps from the sphere at spacetime infinity S_3 to the gauge group G. The possible non-trivial topologies of such maps is determined by $\pi_3(G)$, the equivalence classes of maps from S_3 to G that can be continuously deformed into one another. It has been shown that $\pi_3(SU(N)) = \mathbb{Z}$. There are solutions with any integer winding number as determined by their topological properties at spacetime infinity.

The winding number ν of a given solution can be determined by calculating the Cartan-Maurer integral invariant. It can be shown (maybe during one of the topology series of journal club talks) that the invariant is directly related to the integral of $F\tilde{F}$ over all of spacetime. The relevant result is that

$$\int d^4x F^{a\mu\nu} \tilde{F}^a_{\mu\nu} = \frac{32\pi^2}{g^2}\nu. \tag{5}$$

If we integrate (3), assuming that the current vanishes sufficiently fast at infinity, that there is a $B+L$ charge difference between the states in the distant past and future. The change in $B+L$ is given by $2n_F\nu$. We thus see the connection between anomalies and winding number: the amount of violation of an anomalous symmetry when scattering off some extended field configuration is proportional to the winding number of the field configuration. This sort of scattering is exactly what we consider when we work with instantons.

We can go one step further at this stage. We can define a topological current (in Euclidean spacetime, if we are being careful) associated with a field configuration that has non-trivial topology:

$$j_T^\mu = \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \left(A_\nu^a F_{\rho\sigma}^a - \frac{1}{3} f^{abc} A_\nu^a A_\rho^b A_\sigma^c \right). \tag{6}$$

The divergence of this current can be shown to be proportional to $F\tilde{F}$. Thus, a field configuration can effectively carry charge under $B+L$ so that the *total* current $j = j_{B+L} + j_T$ is conserved. Thus, for a field configuration at a specific time, we can associate an effective $B+L$ charge

$$Q_{B+L} = \int d^3x j_T^0. \tag{7}$$

Thus, we see that even for the static configurations that we will consider when discussing thermodynamics, there is a connection between topological charge and anomalies. The charge of the configuration need not be that of a particular allowed vacuum. If it is not, however, then the solution cannot be stable. Unstable solutions that have $B+L$ charge interpolating between allowed vacua are called sphalerons.

We will discuss the connection between anomalies and winding number in more detail later after we introduce both sphalerons and instantons in more detail. The goal here was to review the necessary anomaly technology and to hint at the upcoming connections to topology. It should make sense that there is a connection between topology and anomalies since the processes that mediate anomalous violation of symmetries must be non-perturbative and so involve extended field configurations. We have made the relation a little more crisp and we will clarify further once we have discussed the topological solutions we are interested in and why they are relevant.

III. BEYOND PT: EXTENDED FIELD CONFIGURATIONS OF $SU(2)$

Let us now focus on $G = SU(2)$. Much of what we say will be extensible to other gauge groups. The concrete solutions that we will write down will, however, be specific to this gauge group.

We are looking for solutions to the equations of motion that are pure gauge at infinity. A potential is pure gauge if and only if it can be written in the form

$$A_\mu = g^{-1}\partial_\mu g, \tag{8}$$

where $g \in SU(2)$ is any local gauge transformation. We will stick to a notation where we write A^μ as an anti-Hermitian 2×2 matrix. To recover the "usual" way of writing A, we can always take $i\mathrm{tr}(A^\mu\sigma^a)$. We will thus look for solutions of the form

$$A_\mu = f(r^2)g^{-1}\partial_\mu g, \tag{9}$$

where r will be the norm of either the space or spacetime coordinate depending on the context. We want to find $f(r)$ such that $f(\infty) = 1$. In other words, the solution is pure gauge at infinity. We would also like to avoid singularities, so we look for solutions with $f(0) = 0$.

To get a bit more concrete, we would first like to find the winding number $\nu = 1$ instanton for a pure $SU(2)$ Yang-Mills theory. The point that makes $SU(2)$ easy that it is topologically equivalent to S^3. A point on S^3 can be described by a unit vector in $\mathbb{R}^4$ labeled by coordinates $\hat{x} = (x_1, x_2, x_3, x_4)/r$. The point x/r on the sphere is indentified with the $SU(2)$ gauge transformation

$$\frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{r}. \tag{10}$$

One can show that any element of $SU(2)$ can be written in this form.

The simplest mapping from S^3 to $SU(2)$ is the trivial map:

$$g^{(0)}(x) = 1. \tag{11}$$

This mapping has trivial topology. The simplest map with non-trivial topology is the identity map

$$g^{(1)}(x) = \frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{r}, \tag{12}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$. By calculating the Maurer-Cartan integral invariant, one can verify that this mapping has winding number $\nu = 1$. The higher winding number solutions are given by higher powers of $g^{(1)}$:

$$g^{(\nu)}(x) = [g^{(1)}]^\nu. \tag{13}$$

To find the instanton, we want a solution of the form (9), with $g = g^{(1)}$. For the solution to be called an instanton, it must minimize the Euclidean action. When we discuss instantons, we will see that there are very good reasons for this requirement. In particular, the Euclidean action should be finite so that we can do a valid perturbation theory about this solution. To minimize the Euclidean action in the easiest way, we will need to do a little bit of math trickery. First, let us define a notational convention called the Cartan inner product:

$$(F, G) = -2\mathrm{tr}(FG), \tag{14}$$

where F and G are fields that transform in the adjoint (plus, possibly, some extra gauge term). Recall that we are using a notation where such fields are represented by anti-Hermitian matrices. The Yang-Mills Euclidean action is then

$$S_E = \frac{1}{4g^2}\int d^4x(F,F) = \frac{1}{4g^2}\int d^4x(\tilde{F},\tilde{F}), \tag{15}$$

where we rescale F to pull out the gauge coupling in front. The integral can be bounded using the Schwartz inequality

$$S_E = \left[\int d^4x(F,F)\int d^4x(\tilde{F},\tilde{F})\right]^{1/2} \geq |\int d^4x(F,\tilde{F})| = 32\pi^2\nu, \tag{16}$$

where we use the relation (5) in the last step. The action is then bounded from below by $S_E \geq 8\pi^2/g^2\nu$ and the bound is saturated when $F = \tilde{F}$. We thus look for solutions with $F = \tilde{F}$. This is a first order differential equation for f, which yields the solution

$$f(r^2) = \frac{r^2}{r^2 + R^2}. \tag{17}$$

We have thus constructed an instanton solution with $\nu = 1$. The constant R is called the size of the instanton. This instanton interpolates between vacua with $\Delta Q_{B+L} = 2n_f$.

When we add a Higgs doublet, the situation becomes more complicated. The instanton solution in this case can no longer be determined analytically. We can say that the instanton solution for the gauge field still has the form (9). There is, however, now a natural upper bound on the size of the instanton given by the weak scale $(gv)^{-1}$. At larger distances, electroweak symmetry is broken down to electromagnetism. Under electromagnetism alone, neither baryon number nor lepton number is anomalous. This fact will be important later.

Finally, we would like to determine the sphaleron solution in the Higgsed theory. This solution is a static, unstable solution. By unstable, we mean that it sits at a saddle point of the potential energy functional. It is, essentially, the solution of minimum energy that sits at the peak of the potential barrier between vacua of different $B + L$. We will not prove that this solution has these property. We will make an intuitive argument for why this makes sense. We write down the solution in a gauge where $A_0 = A_r = 0$. There is then a solution of the form

$$A_i = -2ivf(gvr)(g^{(1)})^{-1}\partial_i g^{(1)}, \qquad H = \frac{v}{\sqrt{2}}h(gvr)g^{(1)}\begin{pmatrix}0\\1\end{pmatrix}, \tag{18}$$

with x_0 set to 0. There is a solution of exactly the same form as the instanton solution, but "made static." This is no conincidence. What we have done is taken the static portion of the "largest" instanton solution at exactly the center of the instanton.

The sphaleron has a few other properties that we will not show, but will mention. Its effective baryon charge, as defined by (7) is $Q_{B+L} = n_f$, half way between the initial and final $B + L$ charges. It has a finite energy given by

$$E = \left(\frac{2M_W}{\alpha_W}\right)\overline{E}, \qquad 1.56 < \overline{E} < 2.72, \tag{19}$$

depending on the size of the Higgs quartic coupling λ. The radius of the sphaleron is of order $(2M_W)^{-1}$.

IV. SPHALERONS AND FINITE-TEMPERATURE FLUCTUATIONS

The goal of this section is to understand finite-temperature transitions between vacua with different $B + L$ charge. In doing so, we will make use of the sphaleron solution found in the previous section. In order to get there, we first study a toy example that exhibits all of the qualitative features we would like to study without the computational difficiulties.

Consider a pendulum in a potential symmetric about a minimum at $\theta = 0$ as illustrated in Figure 1. There are essentially two scales in this system: the frequency of expansions about $\theta = 0$, $\omega_0 \equiv V''(0)$, and the height of the potential barrier from $\theta = 0$ to $\theta = \pi$, $V_0 \equiv V(\pi) - V(0)$. We will take $V(0) = 0$ for simplicity. Consider the system to be at temperature T. There are three temperature regimes of interest.

The first is $T \ll \omega_0$. This is the zero-temperature, purely quantum limit. In this limit, we need to use instantons to calculate transitions and we will discuss such transitions later. The second is $\omega_0 \ll T \ll V_0$. In this regime, many quanta are excited and the system is classical. The transitions between vacua, however, are expected to have

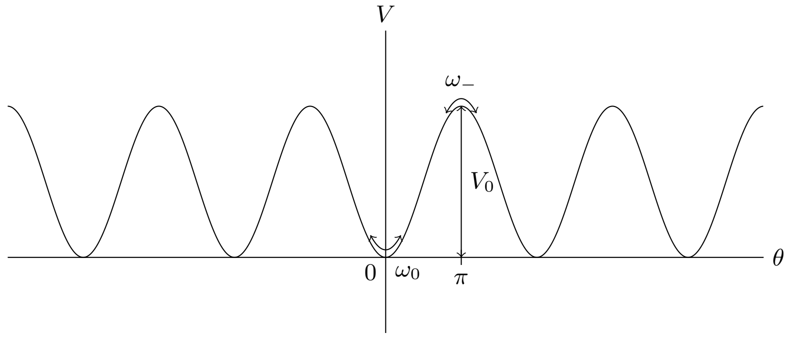

FIG. 1. Potential energy of a pendulum in a symmetric potential.

a Boltzmann suppression $e^{-\beta V_0}$. The third is $T \gg V_0$. In this regime, transitions between vacua occur with order 1 probability. The scale that sets the rate is T, as that is the only relevant scale.

We would like to study our system in a little more detail quantatively. First, let us consider the limit $\omega_0 \ll T \ll V_0$. This limit will turn out to be extremely important. We would like to make our statement that transitions occur with a rate suppressed by $e^{-\beta V_0}$ more crisp and come up with a general expression that we can extend to more complicated system. The rate of transitions is given exactly by calculating the probability that the pendulum is found at $\theta = \pi$ moving in the positive θ direction multiplied by the rate at which the pendulum in such a state crosses the barrier:

$$\begin{aligned}\Gamma &= \langle \delta(\theta-\pi) p \theta(p) \rangle \\ &= \frac{\int dp d\theta \exp\{-\beta \left(\frac{1}{2}p^2 + V(\theta)\right)\} \delta(\theta-\pi) p\theta(p)}{\int dp d\theta \exp\{-\beta \left(\frac{1}{2}p^2 + V(\theta)\right)}\}.\end{aligned} \tag{20}$$

The integration of the numererator is trivial: the momentum integral is Gaussian and the angle integration is over a delta function. In general, we cannot perform the integration in the denominator without making some approximations. In the limit we are considering, the particle will mostly be found near the bottom of the potential well, which is at $\theta = 0$. We can thus expand the potential to quadratic order about $\theta = 0$, leading to a Gaussian integral. We can further make the approximation of sending the angular integration limits to $\pm\infty$, so that the integration can be performed analytically. The result is that

$$\Gamma \approx \frac{\omega_0}{2\pi} e^{-\beta V_0}, \tag{21}$$

where we recover the expected Boltzmann suppression and find the scale of the rate determined by ω_0.

We would like to rewrite this expression in a way that can be generalized easily to field theory. The approach is similar to that when dealing with tunneling out a false vacuum. In that case, the free energy acquires a small imaginary part which is proportional to the rate of transitions. We will skip over this argument, as it is tenuous at best, and simply proceed as follows. Consider the partition function Z expanded about the peak of the potential to quadratic order. We will call this function Z_{sp}. This quantity is not well defined, but we can analytically continue to get an imaginary partition function. Let's study the quantity

$$T \frac{\text{Im} Z_{\text{sp}}}{Z_0}, \tag{22}$$

where Z_0 is the partition function expanded to quadratic order about the vacuum. Suppose that $V''(\pi) = -\omega_-$. Performing the Gaussian integrations in the numerator and denominator, we find

$$T \frac{\text{Im} Z_{\text{sp}}}{Z_0} = \frac{\omega_0}{2\omega_- \beta} e^{-\beta V_0}, \tag{23}$$

where we include a "conventional" factor of 1/2 due to the analytic continuation. Comparing (23) to (21), we find the important relation

$$\Gamma \approx \frac{\omega_-}{\pi} \frac{\text{Im} Z_{\text{sp}}}{Z_0}. \tag{24}$$

What we have learned is that, in the approximation where $\omega_0 \ll T \ll V_0$, the rate can be related to Gaussian expansions of the partition function about an unstable solution at the peak of the potential and about one of the true vacua. The unstable solution is, as we have already seen, called a sphaleron. In this case, the sphaleron solutions is trivial: $\theta(t) = \pi$.

Note that, in the high temperature limit, we can negelct the potential entirely in both the numerator and denominator of (20). The result of this approximation is that the rate is given by

$$\Gamma \approx \sqrt{\frac{T}{8\pi^3}}. \tag{25}$$

That is, the rate is set entirely by the temperature (which in our units has dimensions of time to the power 1/2), up to some numerical coefficients.

This formalism carries over almost wholesale to the study of the $B+L$ anomaly. We have already see that there is a sphaleron solution with $E \sim 2M_W/\alpha_W$. This energy corresponds to our V_0. Furthermore, the excitations about the minimum of the potential in the $SU(2)$ direction are electroweak bosons, so that M_W corresponds to our ω_0. We wind up immediately with the result that in the limit of $2M_W \ll T \ll 2M_W/\alpha_W$, the rate is given by

$$\Gamma \sim \frac{M_W(T)}{\pi} e^{-4\beta M_W(T)/\alpha_W(T)}. \tag{26}$$

A more careful argument leads to more or less the same result.

The temperature dependence of this result is important for two reasons. The first is that the electroweak theory has a critical point roughly in this range of temperatures. The second is that in making the approximations we've made, we have made assumptions of weak coupling. The effective thermal coupling constant, however, can get large at certain temperatures. Thus, we take a detour to discuss the temperature dependence of this physics.

We can calculate the renormalization of the Higgs VEV as we go from $T = 0$ to finite T. The renormalized Higgs VEV is found to be

$$v^2(T) = v^2(0) - \left(\frac{1}{2} + \frac{3g^2}{16\lambda}\right) T^2, \tag{27}$$

where g is the electroweak coupling and λ is the Higgs quartic coupling. The critical temperature is the one at which $v^2(T_c) = 0$. Solving, we find that

$$T_c = v(0) \left(\frac{2}{1 + \frac{3}{8}\frac{g^2}{\lambda}}\right)^{1/2}. \tag{28}$$

The corresponding renormalization of the W mass can be written as

$$M_W(T) = M_W(0) \left[1 - \left(\frac{T}{T_c}\right)^2\right]^{1/2}. \tag{29}$$

These results mean that the critical temperature is $\sim 10M_W$, within the realm of our approximation.

Furthermore, we have made the approximation of ignoring the time-dependent modes in calculating Z_{sp}: no dynamics ever entered our approximations. This is the approximation that $gv\beta \ll 1$, which is only valid in the low-temperature limit. In this limit, our Euclidean action gets replace with an effective 3-dimensional action, which has a coupling constant

$$\alpha_3 = \frac{gT}{4\pi v} = \alpha_W \frac{T}{2M_W}. \tag{30}$$

As $T \to T_c$, this coupling becomes large and our perturbative, time-independent approximation breaks down.

In making this estimate, we've glossed over a sublety that can be quite important. Generically, the sphaleron will be symmetric under some subset of the symmetries of the theory. This leads to zero modes in excitations about the sphaleron solution, as discussed during our monopoles camp. We cannot perform a Gaussian integration over such modes and they must be treated seperately. We do not go over this calculation, but merely mention that it gives an enhancement of g_3^{-1} for each zero mode and a factor of the cube of the size of the sphaleron R^3 from translations. In principle, these factors can be calculated using collective coordinates. They have a significant, but coupling dependent effect. We will come back to such effects when we discuss instantons. It is also these effects that guarantee that correlation functions of operators that do not have the proper $B+L$ charge will vanish.

As the temperature increases, our system approaches the critical point of the electroweak phase transition, at which $M_W(T) \to 0$. Thus, we might expect that the rate vanishes as we go to this limit. In this limit, our approximation breaks down and a different approach is necessary. Well above T_c, transitions that pass through the sphaleron configiguration are no longer favored. The sphaleron configuration is the lowest energy configuration over the barrier. At high temperatures, however, the sphaleron size is large and smaller configurations begin to contribute. These smaller configurations are favored by entropy as more states become available at higher temperatures. The typical size of the contributing configurations is determined by requiring that the Boltzmann factor be order 1. Larger (lower energy) configurations are disfavored by entropy, while smaller configurations (higher energy) are disfavored by energy and are Boltzmann suppressed. We thus expect solutions of energy $E \sim T$, which have size $(\alpha_W T)^{-1}$ as the scale of the solutions is suppressed compared to the scale of their energies. The rate of baryon number violating processes is then no exponentially suppressed, but will still generally be suppressed by some powers of α_W due to the contribution of enhancement of the size of the solutions relative to T^{-1} to the translational symmetry factor, giving a rate

$$\Gamma \sim \alpha_W^n T. \tag{31}$$

There is one more sublety of this whole subject that we would like to address at this point: the relation between sphalerons and instantons. We have already mentioned that amplitudes for $B+L$ violating processes are suppesed by $e^{-8\pi^2/g^2}$. Even at high temperatures, this would seem to make the rate for anomalous $B+L$ violation vanishingly small. There are many subleties when dealing with instantons (see "The Sphaleron Strikes Back" for further details), but the most relevant one for our purposes is the fact that the finite temperature transitions do not occur in a vacuum, but rather in a thermal bath of particles.

In particular, we find that while the amplitude for a process like $\langle qqq\ell\rangle$ is highly suppressed, there are unsuppressed amplitudes for $\langle qqq\ell W^n\rangle$ for $n \sim \pi/\alpha_W$. Such a process would require an extremely high energy to produce at zero temperature due to phase space suppression. At sufficiently high temperature, there is a bath of W's available to mediate the process.

To see how this works explicitly, we go through an extremely simple, but instructive toy example. Consider a 0-dimensional theory. That is, the path integral is over a single variable. We take the action to be

$$S(x) = \frac{1}{g^2}\left[1 + \left(x - \frac{1}{g}\right)^2\right]. \tag{32}$$

This action has a minimum at $x_0 = g^{-1}$, with the action given by $S_{\min} = g^{-2}$. We would like to analyse the $2n$-point "correlation function." It is defined by the integral

$$I_n = \int dx e^{-S(x)} x^{2n}. \tag{33}$$

The integration can be done analytically using Mathematica and the leading dependence in the small coupling regime is

$$I_n \approx \sqrt{\pi} g^{1-2n} e^{-1/g^2}. \tag{34}$$

For small n, the exponential suppression dominates and the correlation function vanishes. For $n \approx g^{-2}$, we find that

$$I_n \sim (g^{-2})^{g^{-2}} e^{-g^{-2}} \approx (g^{-2})!, \tag{35}$$

using the Stirling approximation in the last step. Thus, the large n amplitude is by no means suppressed. It is, in fact, very big.

We find that the naive instanton calculation becomes problematic when there are a large number of quanta. There are further issues with the instanton estimate, including momentum dependence of the amplitude, the continuation to real time, and the phase space integration. We learn that in the regime with many quanta, that is the high temperature limit, we should not rely on instantons, but rather on thermal calculations using sphalerons. Hence, the return of the quanta ensures the importance of sphalerons for thermal calculations!

V. QUANTUM TUNNELING AT ZERO TEMPERATURE

Quantum tunneling at zero temperature can be estimated in the semi-classical limit using instantons. There is a sublety with this calculation that has, to the best of my knowledge, not been fully resolved. Recall that the instanton was a solution to the equations of motion of the *Euclidean* action. We will assume throughout this section that Euclidean correlation functions can be analytically continued to Minkowski space.

When calculating the path integral, generally, we can break up the possible gauge field configurations based on their winding number and the values of their parametrizations by the symmetries under which they transform nontrivially:

$$\langle \mathcal{O} \rangle = \mathcal{N} \sum_{\nu} \int d\lambda_\nu \mathcal{D} A_{\nu,\lambda_\nu} \mathcal{O} e^{-S_E[A_{\nu,\lambda_\nu}]}, \tag{36}$$

where λ_ν are the parameters that describe solutions with winding number ν and A_{ν,λ_ν} are all field configurations with winding number ν with symmetry parameter λ_ν. $\mathcal{N}$ is a normalization constant. Note that the integration over the coordinates λ_ν are superfluous. Excitations in these directions are zero modes and lead to a singular equation for the Green's function. Just as when we deal with gauge degrees of freedom, we have to use a Fadeev-Popov trick to get a well-defined path integral and "fix" these degrees of freedom. This is the method of collective coordinates. We do not discuss this procedure further, but it does have extremely important consequences for the calculation of amplitudes. In particular, it gives a g^{-8} enhancement, among other factors.

We can then expand the action $A_{\nu,\lambda_\nu} = A_{\text{inst},\nu,\lambda_\nu} + \tilde{A}_{\nu,\lambda_\nu}$. As we are expanding about the classical instanton solution, we get an expression like

$$S = S_{\text{inst}} + \int d^4x d^4y \frac{\delta^2 S}{\delta A(x) \delta A(y)} |_{A=A_{\text{inst}}} \tilde{A}(x) \tilde{A}(y) + \ldots, \tag{37}$$

where we write $A = A_{\text{inst}} + \tilde{A}$. The linear term vanishes since A_{inst} is a solution to the classical equations of motion. Notice that $S_{\text{inst}} = 8\pi^2\nu/g^2$ by (5). Thus, an amplitude involving a winding number 1 instanton will be automatically be suppressed by $e^{-8\pi^2/g^2}$ as mentioned earlier.

We should also perform the expansion with respect to other fields in the theory. In particular, we should expand with respect to the fermions of the theory:

$$S \supset \int d^4x d^4y \frac{\delta^2 S}{\delta f_i(x) \delta f_j(x)} |_{A=A_{\text{inst}}} f_i(x) f_j(x) \tag{38}$$

If $\mathcal{O}$ does not contain the fields f_i and f_j, then the path integral will yield a factor of

$$\det \frac{\delta^2 S}{\delta f_i(x) \delta f_j(x)} |_{A=A_{\text{inst}}}. \tag{39}$$

If there are any zero modes, then the path integral vanishes. In fact, there will be zero modes for all the fermions involved in the anomaly loop. This is just telling us that instantons mediate $B+L$ violation according to the anomaly (3). There cannot be vacuum-to-vacuum transitions as this violates the combined topological and $B+L$ charge. Thus, the operator must include *all* of the field involved in the anomaly loop. We have now seen this fact in two different ways.

The correlation function for $B+L$ violation with N_f flavors was performed in the case with out a Higgs and with massless fermions by 't Hooft. There are three generations of quarks with three colors each and three generations of leptons. The effecive operator must include all of these fields, as we have now argued in two different ways. The result is that

$$\mathcal{L} = \frac{2^{32}\pi^{18}}{48} g^{-8} e^{-\frac{8\pi^2}{g^2}} e^{A-6B} \frac{\mu^{10/3}}{(gv)^{43/3}} \det f_i f_j, \tag{40}$$

where the determinant runs over the possible $SU(2)$ doublet chiral fermions $f_i = \{\ell_I, q_I^a\}$, μ is the scale of the interaction, gv is used as in IR cutoff in the integration over instanton sizes, and A and B are numerical constants that are order 1. We write this full expression for the effective operator of $B+L$ violation to highlight several points. We point out that the scale associated with the interactions is the associated with some IR cutoff, which is of order the sphaleron size gv in this case. This cutoff makes sense as the sphaleron is the lowest energy excitation that crosses the potential barrier between different $B+L$ vacua. We also highlight the extreme suppression of the dimensionless coupling that comes with the operator. Numerically, for $\mu \sim 1$ Tev, the scale associate with this operator is $\sim 10^3$ TeV, so it is highly suppressed.

VI. CONSEQUENCES AND CONCLUSIONS

In this review, we have focused on computational aspects of $B+L$ violating processes. We have neglected, up to this point, several other important aspects of these processes.

We have not discussed in great detail the topology necessary to understand the topological solutions involved in $B+L$ violation. It would be interesting to study on more general grounds the topological consequences of instantons. We leave this for other talks in the Journal Club.

We have not explicitly calculated explicitly any rates or amplitudes for $B+L$ violation using sphalerons or instantons. These calculations are technically demanding. They require a full understanding of collective coordinates as well as a very careful approach to regularization of divergent integrals. It would be interesting to understand the subleties involved in these calculations, if not the whole procedure followed.

Lastly, we have neglected up until this point the consequences of the results derived for the phenomenology SM. We were motivated to study sphalerons by their importance in cosmological bayron and lepton number violation. In particular, they are essential for calculating the residual baryon and lepton asymmetry in the present day universe. If some high-scale physics generates a bayon or lepton number asymmetry, then sphalerons can transfer the asymmetry between the two sectors. As we can now understand, these processes freeze out below $T \sim 2M_W$, leaving a baryon and lepton asymmetry at their equilibrium values at that temperature. Calculating these residual asymmetries involves the interplay of the chemical potentials for the various particles of the SM, as described in my talk on Asymmetric Dark Matter. This is perhaps the most important legacy of anomalous $B+L$ violation for present day phenomenology.

Through this review, we have studied the connection between topics that seemed disconnected a priori: anomalies, non-perturbative physics, and topology. We have seen just what extendended field configurations *do* concretely to modify our approach to field theory calculations. Hopefully, we have come out of this with a deeper understanding of field theory and its intricacies. Even though the proesses that we have seen have a fairly limited range of phenomenological importance, we believe that they are conceptually essential to our picture of field theory.

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