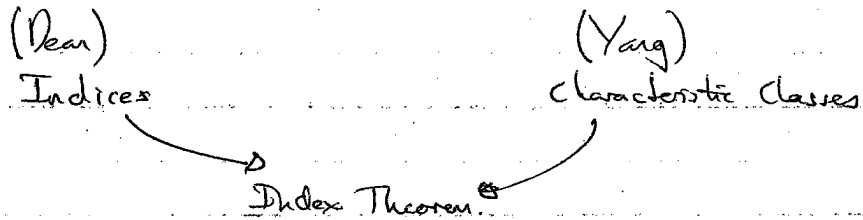


(A prelude)

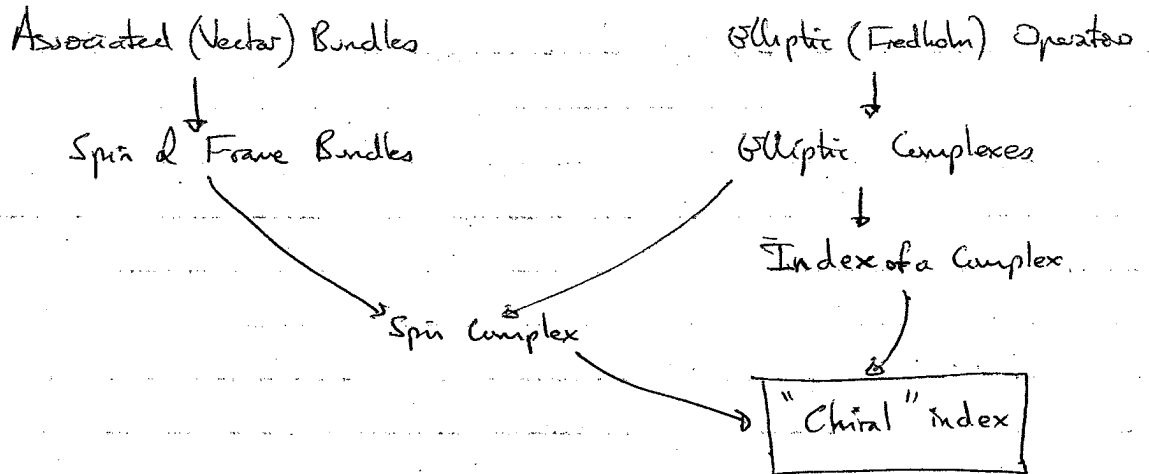
On the Road to Index Theorems

Index theorems reveal a deep connection between the analytic (index) and topological (characteristic class) ^{properties} of something called an (elliptic) complex. They are particularly important in the analysis of anomalies.

Overall Plan:



Plan Today:



Associated Fibre Bundles

Let's begin with a "map up" operation following from Marisa's talk last month. Associate fibre (vector) bundles are the objects on which matter fields are defined, so they are important to discuss!

Consider a principal bundle $P(M, G)$ (or more carefully, (P, π, M, G)), and some manifold F on which the group action of G is well-defined: abstractly

$$g \in G: f \mapsto g \cdot f, \quad f \in F$$

with associativity $gh \cdot f = g \cdot (h \cdot f)$, and $1 \cdot f = f \quad \forall f$.

Def: (Associated Fibre Bundle)

Let (P, π, M, G) be a principal bundle, and F a manifold with well defined group action. Then $(P \times_G F, \pi_F, M, F, G)$ is an associated fibre bundle over M , with projection

$$\pi_F([u, f]_G) = \pi(u)$$

Lemma: $(P \times_G F, \pi_F, M, F, G)$ is a fibre bundle

Proof: To check this, first note that for $[u_1, f_1]_G = [u_2, f_2]_G$, then $(u_1, f_1) \sim (u_2, f_2)$ which implies $\exists g \in G$ s.t.

$$(u_1, f_1) = (u_2 g, g^{-1} f_2)$$

Then

$$\pi_F([u_2 g, g^{-1} f_2]_G) = \pi(u_2) = \pi(u_2 g) = \pi(u_1) = \pi_F([u_1, f_1]_G)$$

so π_F is a well-defined projection on the orbits. It remains to show that $\forall p \in M$,

$$\pi_F^{-1}(p) \cong F.$$

↑ preimage ← a homeomorphism; objects are not groups!

Fix $u \in P$. Now, note first that $f_1 \neq f_2 \Rightarrow (u, f_1) \not\sim (u, f_2)$. Conversely, $(u, f_1) \sim (u, f_2) \Rightarrow \exists g$ s.t. $u = u g \Rightarrow g = 1$ (action is free) $\Rightarrow f_1 = f_2$. Hence

This means $f \mapsto [u, f]$ is 1-1.

$f_1 \neq f_2 \Rightarrow [u, f_1]_G \neq [u, f_2]_G$ & disjoint. Also, note that $(u g, f) \sim (u, g \cdot f)$ is right action on P , which moves along the principal fibre G , is the same as left group action on F ! With these facts in mind, then for $\pi(u) = p$ we have clearly

$$\begin{aligned} \pi_F^{-1}(p) &= \bigsqcup_{f \in F} \{ [u, f]_G \} \cup \{ [u g, f]_G \}_{g \in G} \\ &= \{ [u, f]_G \}_{f \in F} \cup \{ [u, g \cdot f]_G \}_{g \in G, f \in F} \\ &= \{ [u, f]_G \}_{f \in F} \\ &\cong F \end{aligned}$$

under a natural homeomorphism. \square

Now consider two local trivializations ψ_i on the associated vector bundle

$$\psi_i: U_i \times F \rightarrow \pi_F^{-1}(U_i)$$

On $U_i \cap U_j \subset M$ we have a natural local trivialization associated with ϕ_i

$$\psi_i(p, f) = [\phi_i(p, 1), f]$$

But then

$$\begin{aligned} \psi_j(p, f) &= [\phi_i(p, t_{ij}(p)), f] \\ &= [\phi_i(p, 1) + t_{ij}(p), f] \\ &= [\phi_i(p, 1), D_{ij}(p)f] \end{aligned}$$

So we must have structure functions s.t.

$$\psi_i(p, f) = \psi_j(p, D^{-1}(t_{ij}(p))f)$$

That is, the structure group is indeed G , with structure functions the appropriate repⁿ of G .

Spin Bundles

A spin bundle is an associated vector bundle defined by spinor representation of G , ~~of course, this is a spin representation~~. If $G \cong \text{spin}(k)$, then such a repⁿ obviously exists.

Here "lift" refers to consistency constraints of a particular type

A physically more relevant case is $G \cong \text{SO}(k)$. In this case an associated spin bundle exists if the principal (frame) bundle "lifts" to one with $G \cong \text{spin}(k)$.

(Whether this can occur is determined by the 2nd Stiefel-Whitney class; see Yang's talk.)

LM

Eg: Consider a principal (frame) bundle, with structure group $\text{SO}^+(3, 1)$, whose double cover is $\text{SL}(2, \mathbb{C})$. If LM lifts to $P(M, \text{SL}(2, \mathbb{C}))$, then there are two associated spin bundles

$$(W^{\bar{r}}, M, \pi, \overline{\text{SL}(2, \mathbb{C})}, \mathbb{C}^2)$$

where \bar{r} is the $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ reps of $\text{SL}(2, \mathbb{C})$. A section of this is a Weyl spinor.

A section of

$$(D, M, \pi, \text{SL}(2, \mathbb{C}), \mathbb{C}^4)$$

with group action defined by $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ repⁿ is a Dirac spinor.

This yields, as expected,

$$(\Delta s)(x) = \sum_i \frac{\partial^2}{(\partial x^i)^2} s(x).$$

2) Dirac Operator

Let \mathcal{S} be a spin bundle over M , and

$$D: \Gamma(M, \mathcal{S}) \rightarrow \Gamma(M, \mathcal{S}).$$

For $N=1$,

$$(A^K)^\alpha_\beta = \begin{cases} i(\gamma^K)^\alpha_\beta & , K = (0, \dots, 1, \dots, 0) \\ m\delta^\alpha_\beta & , K = 0 \end{cases}$$

we have

$$(D\psi)^\alpha(x) = i(\gamma^{M^i})^\alpha_\beta \partial_\mu \psi^\beta(x) + m\psi^\alpha(x)$$

So both Δ and Dirac operator are differential operators in this sense.

Def¹: The symbol of diff operator D is a matrix

$$\sigma(D, \xi) = \sum_{|K|=N} (A^K)^\alpha_\beta \xi^K$$

where ξ is a real n tuple, $\xi = (\xi_1, \dots, \xi_n)$ and

$$\xi^K = \prod_{\mu} (\xi_\mu)^{m_\mu}, \quad K = (\mu_i)_{i=1}^n$$

Eg:
$$\begin{aligned} \sigma(\Delta, \xi) &= \xi_{(2,0,\dots)} + \xi_{(0,2,\dots)} + \dots \\ &= \sum_{\mu} (\xi_\mu)^2 \end{aligned}$$

We can also define the symbol in a trivialization independent way

Def²: Let $(\mathcal{S}, \pi, M, \mathbb{C}^k)$ be a complex vector bundle, with $\xi \in T(M, \mathbb{R})$, $p \in M$ s.t.

$$\tilde{\xi}(p) = u$$

Let $f \in \mathcal{F}(M)$ s.t. $f(p) = 0$ and define $\xi \equiv df(p) \in T_p^*M$.

The symbol of $D: \Gamma(M, \mathcal{S}) \rightarrow \Gamma(M, \mathcal{S})$ is a map

$$\sigma(D, \xi): E \rightarrow T(M, \mathbb{C}^k)$$

Defⁿ: An elliptic operator with $\dim(\ker(D)) < \infty$, ~~and~~ $\dim(\text{coker}(D)) < \infty$ is a Fredholm operator.

Thm: All elliptic operators on a compact manifold M are Fredholm.

Defⁿ: (Analytical Index) For D Fredholm,

$$\text{ind}(D) \equiv \dim \ker(D) - \dim \text{coker}(D)$$

If we define metrics $\langle \cdot, \cdot \rangle_S$ and $\langle \cdot, \cdot \rangle_F$ on \mathcal{S} & \mathcal{F} , then the adjoint of D is defined via

$$\langle s', Ds \rangle_F = \langle D^*s', s \rangle_S$$

One can show that $\dim \ker(D^*) = \dim \text{coker } D$, in which case

$$\text{ind}(D) = \dim(\ker(D)) - \dim \ker(D^*)$$

This is our first encounter with an index, a analytical property of a Fredholm operator on a complex vector bundle. It turns out that these indices have a topological meaning too: this is the index theorem meaning.

Let's now generalize the concept of an index.

Elliptic Complexes

Defⁿ: An elliptic complex (\mathcal{E}, D) is a nilpotent sequence of Fredholm operators on vector bundles

$$\left(\begin{array}{ccccccc} \dots & \longrightarrow & T(M, \mathcal{E}_{i-1}) & \xrightarrow{D_{i-1}} & T(M, \mathcal{E}_i) & \xrightarrow{D_i} & T(M, \mathcal{E}_{i+1}) & \xrightarrow{D_{i+1}} & \dots \\ & & \longleftarrow & & \longleftarrow & & \longleftarrow & & \\ & & & & D_{i-1}^+ & & D_i^+ & & D_{i+1}^+ \end{array} \right)$$

By nilpotent sequence, is meant $D_i \circ D_{i-1} = 0 \ \forall i$, which implies that

$$\ker(D_i) \supseteq \text{Im}(D_{i-1})$$

(Note that exact sequences are special cases of nilpotent ones)

A convenient notation for the general case: rolled up complexes. Consider an elliptic complex (\mathcal{E}, D) . We can write the indices as

$$\dots \Gamma[M, \mathcal{E}_{2r}] \xrightleftharpoons[D_{2r}^+]{D_{2r}} \Gamma[M, \mathcal{E}_{2r+1}] \xrightleftharpoons[D_{2r+1}^+]{D_{2r+1}} \Gamma[M, \mathcal{E}_{2r+2}] \dots$$

Note that

$$D_{2r} \oplus D_{2r-1}^+ : \Gamma[M, \mathcal{E}_{2r}] \rightarrow \Gamma[M, \mathcal{E}_{2r-1} \oplus \mathcal{E}_{2r+1}]$$

$$D_{2r+1} \oplus D_{2r}^+ : \Gamma[M, \mathcal{E}_{2r+1}] \rightarrow \Gamma[M, \mathcal{E}_{2r} \oplus \mathcal{E}_{2r+2}]$$

We can therefore define an odd and even bundle

$$\mathcal{E}_- \equiv \bigoplus_r \mathcal{E}_{2r+1}, \quad \mathcal{E}_+ \equiv \bigoplus_r \mathcal{E}_{2r}$$

and operator

$$A = \bigoplus_r (D_{2r} \oplus D_{2r-1}^+) : \mathcal{E}_+ \rightarrow \mathcal{E}_-$$

$$A^+ = \bigoplus_r (D_{2r}^+ \oplus D_{2r+1}) : \mathcal{E}_- \rightarrow \mathcal{E}_+$$

We then obtain a rolled up elliptic complex

$$\Gamma[M, \mathcal{E}_+] \begin{array}{c} \xrightarrow{A} \\ \xleftarrow{A^+} \end{array} \Gamma[M, \mathcal{E}_-]$$

The Laplacian

$$\begin{aligned} \Delta_+ &= A^+ A \\ &= \bigoplus_r (D_{2r-1}^+ D_{2r-1} + D_{2r}^+ D_{2r}) \equiv \bigoplus_r \Delta_{2r} \\ \Delta_- &= A A^+ \\ &= \bigoplus_r \Delta_{2r+1} \quad \text{similarity.} \end{aligned}$$

Then

$$\begin{aligned} \text{ind}(\mathcal{E}_\pm, A) &= \dim \ker \Delta_+ - \dim \ker \Delta_- \\ &= \sum_r \dim \ker \Delta_r (-1)^r \\ &= \text{ind}(\mathcal{E}, D) \end{aligned}$$

So any elliptic complex is equivalent to a two term rolled up complex.]

We can choose a basis of the fibre such that

$$\gamma^{2l+1} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

and then write the fibre $\mathbb{C}^{2l} = \mathbb{C}^+ \oplus \mathbb{C}^-$. As a consequence, we can decompose the spin bundle into two associated spin bundles, on which $\gamma^{2l+1} = \pm 1$ on the fibres. We can then decompose the sections (using the above notation $s: x \rightarrow s^a(x)$) as

$$\Delta(M) = \Delta^+(M) \oplus \Delta^-(M).$$

So

$$\begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} \in \Delta^+(M), \quad \begin{pmatrix} 0 \\ \psi^- \end{pmatrix} \in \Delta^-(M).$$

Now, let's find an ^{elliptic} operator acting between $\Delta^\pm(M)$.

Def: For $\psi \in \Delta(M)$, the Dirac operator in curved space

$$i\not{D}\psi = i\gamma^m \nabla_m \psi = i\gamma^m \left(\partial_m + i\omega_m^{\alpha\beta} \sum_{\alpha < \beta} \alpha_\beta \right) \psi$$

\uparrow spin connection

Lemma: $i\not{D}$ is elliptic.

Proof: For $f \in \mathcal{C}^\infty(M)$, $p \in M$, $\tilde{\psi} \in \Delta(M)$ s.t. $f(p) = 0$, $\tilde{\psi}(p) = \psi$, $i\not{D}f = i\not{D}\tilde{\psi}$ we have

symbol

$$\begin{aligned} \sigma(i\not{D}, \xi) \psi &= i\not{D}(f\tilde{\psi})(p) \\ &= (i\not{D}f)_p \psi + i f(p) (i\not{D}\tilde{\psi})(p) \\ &= i\not{D}\psi \end{aligned}$$

Now, $\not{D}\not{D} = \xi^2$, so $\sigma(i\not{D}, \xi)$ is invertible, as $\sigma(i\not{D}, \xi)^2 = \xi^2 \neq 0$ if $\xi \neq 0$. Hence $i\not{D}$ is elliptic. \blacksquare

In the basis for which γ^{2l+1} is diagonal, we can show

$$\gamma^{2l+1} = \begin{pmatrix} 0 & i\alpha^{2l} \\ -i\alpha^{2l} & 0 \end{pmatrix}, \quad \alpha^p = 1, \dots, 2l$$

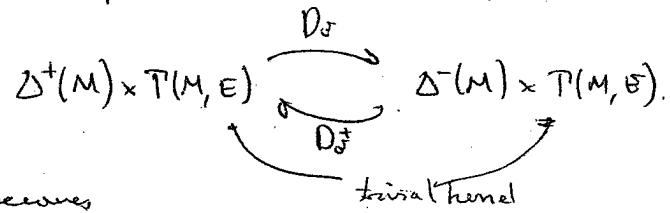
so that

$$i\not{D} = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}$$

We call $P_3 \times \mathcal{E}$ a twisted spin bundle. The Dirac operator

$$D_{\mathcal{E}} = i \gamma^{\mu} (\partial_{\mu} + \omega_{\mu} + A_{\mu})$$

is an elliptic operator that produces a twisted spin complex



The index theorem becomes

$$\eta_+ - \eta_- = \int_M \hat{A}(TM) \text{ch}(\mathcal{E})|_{\text{vol.}}$$