(A prelude)
On the Road to Index Theorems

Index theorems reveal a deep connection between the analytic (index) and topological (characteristic class) properties of something called an (elliptic) complex. They are particularly important in the analysis of anomalies.

Overall Plan:

$$
\begin{tikzcd}
\text{(Dean) Indices} \arrow[dr] & & \text{(Yang) Characteristic Classes} \arrow[dl] \\
& \text{Index Theorem} &
\end{tikzcd}
$$

Plan Today:

$$
\begin{tikzcd}
\text{Associated (Vector) Bundles} \arrow[d] & & \text{Elliptic (Fredholm) Operators} \arrow[d] \\
\text{Spin \& Frame Bundles} \arrow[dr] & & \text{Elliptic Complexes} \arrow[dl] \arrow[d] \\
& \text{Spin Complex} \arrow[dr] & \text{Index of a Complex} \arrow[d] \\
& & \boxed{\text{"Chiral" index}}
\end{tikzcd}
$$

Associated Fibre Bundles

Let's begin with a "mop up" operation following from Mario's talk last month. Associate fibre (vector) bundles are the objects on which matter fields are defined, so they are important to discuss!

Consider a principal bundle $P(M, G)$ (or more carefully, (P, π, M, G)), and some manifold F on which the group action of G is well-defined: abstractly

$$g \in G : f \mapsto g \cdot f, \quad f \in F.$$

with associativity $gh \cdot f = g \cdot (h \cdot f)$, and $1 \cdot f = f \ \forall f$.

Def$^{\underline{n}}$: (Associated Fibre Bundle)
Let (P, π, M, G) be a principal bundle, and F a manifold with well defined group action.
Then $(P \times_G F, \pi_F, M, F, G)$ is an associated fibre bundle over M, with projection

$$\pi_F([u, f]_G) = \pi(u)$$

Lemma: $(P \times_G F, \pi_F, M, F, G)$ is a fibre bundle
Proof: To check this, first note that for $[u_1, f_1] = [u_2, f_2]$, then $(u_1, f_1) \sim (u_2, f_2)$ which implies $\exists g \in G$ s.t.

$$(u_1, f_1) = (u_2 g, g^{-1} f_2)$$

Then

$$\pi_F([u_2, f_2]) = \pi(u_2) = \pi(u_2 g) = \pi(u_1) = \pi_F([u_1, f_1])$$

so π_F is a well-defined projection on the orbits. It remains to show that $\forall p \in M$,

$$\pi_F^{-1}(p) \simeq F.$$

preimage — a homeomorphism; objects are not groups!

Fix $u \in P$. Now, note first that $f_1 \neq f_2 \Rightarrow (u, f_1) \not\sim (u, f_2)$. Contrapositively, $(u, f_1) \sim (u, f_2) \Rightarrow \exists g$ s.t. $u = ug \Rightarrow g = 1$ (action is free) $\Rightarrow f_1 = f_2$. Hence

$$f_1 \neq f_2 \Rightarrow [u, f_1]_G \neq [u, f_2]_G \text{ \& disjoint.}$$

This means $f \mapsto [u, f]$ is 1-1.

Also, note that $(ug, f) \sim (u, g \cdot f)$ so right action on P, which moves along the principal fibre G, is the same as left group action on F! With these facts in mind, then for $\pi(u) = p$ we have clearly

$$\begin{aligned}
\pi_F^{-1}(p) &= \underbrace{\{[u, f]\}_{f \in F}}_{\text{disjoint}} \cup \{[ug, f]\}_{\substack{g \in G \\ f \in F}} \\
&= \underbrace{\{[u, f]\}_{f \in F}}_{\text{disjoint}} \cup \{[u, g \cdot f]\}_{\substack{g \in G \\ f \in F}} \\
&= \underbrace{\{[u, f]\}_{f \in F}}_{\text{disjoint}} \\
&\simeq F
\end{aligned}$$

under a trivial homeomorphism. ■

Now consider two local trivializations ψ_i on the associated vector bundle

$$\psi_i : U_i \times F \to \pi_F^{-1}(U_i)$$

On $U_i \cap U_j \subset M$ we have a natural local trivialization associated with ϕ_i

$$\psi_i(p,f) = \left[\phi_i(p,1), f\right]$$

But then

$$\begin{aligned}\psi_j(p,f) &= \left[\phi_i(p, t_{ij}(p)), f\right] \\ &= \left[\phi_i(p,1)\, t_{ij}(p), f\right] \\ &= \left[\phi_i(p,1), D(t_{ij}(p)) f\right]\end{aligned}$$

So we must have structure functions s.t.

$$\psi_i(p,f) = \psi_j(p, D^{-1}(t_{ij}(p)) f)$$

That is, the structure group is indeed G, with structure functions the appropriate rep^n of G.

Spin Bundles

A spin bundle is an associated vector bundle defined by a spinor representation of G, ~~Of course, the spin group spin(k)~~. If $G \simeq spin(k)$, then such a rep^n obviously exists. A physically ~~more~~ relevant case is $G \simeq SO(k)$. In this case an associated spin bundle exists if the principal (frame) bundle "lifts" to one with $G \simeq spin(k)$. (Whether this can ~~occur~~ occur is determined by the 2nd Stiefel-Whitney class: see Yang's talk.)

Here "lift" refers to consistency constraints of a particular type

Eg: Consider a principal (frame) bundle LM with structure group $SO^+(3,1)$, whose double cover is $SL(2,\mathbb{C})$. If LM lifts to $P(M, SL(2,\mathbb{C}))$, then there are two associated spin bundles

$$\left(W^{(F)}, M, \pi, \overline{SL(2,\mathbb{C})}, \mathbb{C}^2\right)$$

where F is the "$(\frac{1}{2},0)$", "$(0,\frac{1}{2})$" rep^{n}s of $SL(2,\mathbb{C})$. A section of this is a Weyl spinor.

A section of

$$\left(D, M, \pi, SL(2,\mathbb{C}), \mathbb{C}^4\right)$$

with group action defined by $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ rep^n is a Dirac spinor.

This yields, as expected,

$$(\Delta s)(x) = \sum_i \frac{\partial^2}{(\partial x^i)^2} s(x).$$

2) (Dirac Operator)
Let σ be a spin bundle over M, and

$$D: \Gamma(M,\sigma) \to \Gamma(M,\sigma).$$

For $N=1$,

$$(A^K)^\alpha{}_\beta = \begin{cases} i(\gamma^K)^\alpha{}_\beta & , K = (0,..,1,..0) \\ m\delta^\alpha{}_\beta & , K = 0 \end{cases}$$

we have

$$(D\psi)^\alpha(x) = i(\gamma^\mu)^\alpha{}_\beta \partial_\mu \psi^\beta(x) + m\psi^\alpha(x)$$

So both Δ and Dirac operator are differential operators in this sense.

Def^{n}: The symbol of diff operator D is a matrix

$$\sigma(D,\xi) \equiv \sum_{|K|=N} (A^K)^\alpha{}_a \, \xi_K$$

where ξ is a real m tuple, $\xi = (\xi_1,..,\xi_m)$ and

$$\xi_K = \prod_\mu (\xi_\mu)^{\mu_i}, \quad K = (\mu_i)_{i=1}^m$$

Eg: $\sigma(\Delta,\xi) = \xi_{(2,0..)} + \xi_{(0,2,..)} + \cdots$
$= \sum_\mu (\xi_\mu)^2$

We can also define the symbol in a trivialization independent way
Def^{n}: Let $(\sigma, \pi, M, \mathbb{C}^k)$ be a complex vector bundle, with $\tilde{s} \in \Gamma(M,E)$, $p \in M$ s.t.
$\tilde{s}(p) = u$
Let $f \in \mathcal{F}(M)$ st $f(p) = 0$ and define $\xi \equiv df(p) \in T_p^*M$.
The symbol of $D: \Gamma(M,\sigma) \to \Gamma(M,F)$ is a map

$$\sigma(D,\xi): E \to \Gamma(M,F)$$

<u>Def$^{\underline{n}}$</u>: An elliptic operator with $\dim(\ker(D)) < \infty$, ~~dimk~~ $\dim(\operatorname{coker}(D)) < \infty$ is a <u>Fredholm</u> operator.

<u>Th'm</u>: All elliptic operators on a compact manifold M are Fredholm.

<u>Def$^{\underline{n}}$</u>: (Analytical Index) For D Fredholm,

$$\boxed{\operatorname{ind}(D) \equiv \dim \ker(D) - \dim \operatorname{coker}(D)}.$$

If we define metrices $\langle , \rangle_{\mathcal{E}}$ and $\langle , \rangle_F$ on $\mathcal{E}$ & F, then the adjoint of D is defined via

$$\langle s', Ds \rangle_F = \langle D^+ s', s \rangle_{\mathcal{E}}$$

One can show that $\dim \ker(D^+) = \dim \operatorname{coker} D$, in which case

$$\boxed{\operatorname{ind}(D) = \dim(\ker(D)) - \dim \ker(D^+).}$$

This is our first encounter with an <u>index</u>; an analytical property of a Fredholm operator on a complex vector bundle. It turns out that these indices have a topological meaning too: this is the index theorem meaning.

Let's now generalize the concept of an index.

<u>Elliptic Complexes</u>

<u>Def$^{\underline{n}}$</u>: An <u>elliptic complex</u> $(\mathcal{E}, D)$ is a nilpotent sequence of Fredholm operators on vector bundles

$$\cdots \longrightarrow \Gamma(M, \mathcal{E}_{i-1}) \xrightarrow{D_{i-1}} \Gamma(M, \mathcal{E}_i) \xrightarrow{D_i} \Gamma(M, \mathcal{E}_{i+1}) \xrightarrow{D_{i+1}} \cdots$$

$$\left(\longleftarrow \quad \xleftarrow[D_{i-1}^+]{} \quad \xleftarrow[D_i^+]{} \quad \xleftarrow[D_{i+1}^+]{} \right)$$

By nilpotent sequence, is meant $D_i \circ D_{i-1} = 0 \;\; \forall i$, which implies that

$$\ker(D_i) \supseteq \operatorname{Im}(D_{i-1})$$

(Note that exact sequences are special cases of nilpotent ones)

A convenient notation for the general case: rolled up complexes. Consider an elliptic complex $(\mathcal{E}, D)$. We can write the indices as

$$\cdots \Gamma[M, \mathcal{E}_{2r}] \underset{D_{2r}^{+}}{\overset{D_{2r}}{\rightleftarrows}} \Gamma[M, \mathcal{E}_{2r+1}] \underset{D_{2r+1}^{+}}{\overset{D_{2r+1}}{\rightleftarrows}} \Gamma[M, \mathcal{E}_{2r+2}] \cdots$$

Note that

$$D_{2r} \oplus D_{2r-1}^{+} : \Gamma[M, \mathcal{E}_{2r}] \to \Gamma[M, \mathcal{E}_{2r-1} \oplus \mathcal{E}_{2r+1}]$$

$$D_{2r+1} \oplus D_{2r}^{+} : \Gamma[M, \mathcal{E}_{2r+1}] \to \Gamma[M, \mathcal{E}_{2r} \oplus \mathcal{E}_{2r+2}]$$

We can therefore define an odd and even bundle

$$\mathcal{E}_{-} \equiv \bigoplus_r \mathcal{E}_{2r+1}, \quad \mathcal{E}_{+} \equiv \bigoplus_r \mathcal{E}_{2r}$$

and operator

$$A = \bigoplus_r \left(D_{2r} \oplus D_{2r-1}^{+}\right) : \mathcal{E}_{+} \to \mathcal{E}_{-}$$

$$A^{+} = \bigoplus_r \left(D_{2r}^{+} \oplus D_{2r+1}\right) : \mathcal{E}_{-} \to \mathcal{E}_{+}$$

We then obtain a rolled up elliptic complex

$$\Gamma[M, \mathcal{E}_{+}] \underset{A^{+}}{\overset{A}{\rightleftarrows}} \Gamma[M, \mathcal{E}_{-}]$$

The Laplacian

$$\begin{aligned}
\Delta_{+} &= A^{+}A \\
&= \bigoplus_r \left(D_{2r-1} D_{2r-1}^{+} + D_{2r}^{+} D_{2r}\right) \equiv \bigoplus \Delta_{2r} \\
\Delta_{-} &= AA^{+} \\
&= \bigoplus_r \Delta_{2r+1} \quad \text{similarly.}
\end{aligned}$$

Then

$$\begin{aligned}
\operatorname{ind}(\mathcal{E}^{\pm}, A) &= \dim\ker \Delta_{+} - \dim\ker \Delta_{-} \\
&= \sum_r \dim\ker \Delta_r (-1)^r \\
&= \operatorname{ind}(\mathcal{E}, D)
\end{aligned}$$

So any elliptic complex is equivalent to a two term rolled up complex.

We ~~de~~ can choose a basis of the fibre such that

$$\gamma^{2l+1} = \begin{array}{c} \begin{array}{cc} 2^{l-1} & 2^{l-1} \end{array} \\ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \end{array}$$

and then write the fibre $\mathbb{C}^{2^l} = \mathbb{C}^+ \oplus \mathbb{C}^-$. As a consequence, we can decompose the spin bundle into two associated spin bundles, on which $\gamma^{2l+1} = \pm 1$ on the fibres. We can then decompose the sections (using the abused notation $s : x \to s^a(x)$) as

$$\Delta(M) = \Delta^+(M) \oplus \Delta^-(M).$$

So

$$\begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} \in \Delta^+(M), \quad \begin{pmatrix} 0 \\ \psi^- \end{pmatrix} \in \Delta^-(M).$$

Now, let's find an elliptic operator acting between $\Delta^\pm(M)$.

Def^n: For $\psi \in \Delta(M)$, the Dirac operator in curved space

$$i\nabla\!\!\!/\,\psi = i\gamma^\mu \nabla_\mu \psi = i\gamma^\mu\left(\partial_\mu + i\omega_\mu^{\alpha\beta}\Sigma_{\alpha\beta}\right)$$

spin connection

Lemma: $i\nabla\!\!\!/$ is elliptic.

Proof: For $f \in \mathcal{F}(M)$, $p \in M$, $\tilde{\psi} \in \Delta(M)$ s.t. $f(p) = 0$, $\tilde{\psi}(p) = \psi$, $i\nabla\!\!\!/ f = i\xi\!\!\!/$ we have symbols

$$\begin{aligned} \sigma(i\nabla\!\!\!/, \xi)\psi &= i\nabla\!\!\!/(f\tilde{\psi})(p) \\ &= (i\nabla\!\!\!/ f)_p \psi + i f(p)^0 (\nabla\!\!\!/\tilde{\psi})(p) \\ &= i\xi\!\!\!/\psi \end{aligned}$$

Now, $\xi\!\!\!/\xi\!\!\!/ = \xi^2$, so $\sigma(i\nabla\!\!\!/, \xi)$ is invertible, as $\sigma(i\nabla\!\!\!/, \xi)^2 = \xi^2 \neq 0$ if $\xi \neq 0$. Hence $i\nabla\!\!\!/$ is elliptic ∎

In the basis for which γ^{2l+1} is diagonal, we can show

$$\gamma^\mu = \begin{pmatrix} 0 & i\alpha^\mu \\ -i\alpha_\mu^+ & 0 \end{pmatrix}, \quad \mu = 1, \dots, 2l$$

so that

$$i\nabla\!\!\!/ = \begin{pmatrix} 0 & D^+ \\ D & 0 \end{pmatrix}$$

We call $P_S \times \mathcal{E}$ a <u>twisted spin bundle</u>. The Dirac operator

$$D_\sigma = i\gamma^\mu(\partial_\mu + \omega_\mu + A_\mu)$$

is an elliptic operator that produces a twisted spin complex

$$\begin{tikzcd} \Delta^+(M) \times \Gamma(M, E) \arrow[r, bend left, "D_\sigma"] & \Delta^-(M) \times \Gamma(M, E) \arrow[l, bend left, "D_\sigma^\dagger"] \end{tikzcd}$$

trivial kernel

The index th'm becomes

$$\nu_+ - \nu_- = \int_M \hat{A}(TM)\, ch(E)\big|_{vol}.$$