Notes to

"CONSTRAINTS ON SUSY BREAKING"

Witten, 1984

1. Intro Remarks

- ~~SUSY~~ @ tree-level well understood

  ↳ realistic physics probably needs ~~SUSY~~ via quantum effects (renorm... @ tree)

  → Dynamical ~~SUSY~~!

- PURPOSE: DERIVE CONSTRAINTS ON CONDITIONS UNDER WHICH DYNAMICAL ~~SUSY~~ CAN OCCUR!

  ↳ we will calculate certain quantities which can be calculated reliably in PT, WHICH MUST VANISH for ~~SUSY~~ to be POSSIBLE.

  ⇒ if they do not vanish, NO ~~SUSY~~!

  ↳ These quantities are "topological invariants" of the field theory (particular to the whole theory rather than a particular field configuration.)

2. $Tr(-1)^F$

- **CONSIDER SUSY THEORIES IN FINITE VOLUME**

  ↳ discrete spectrum of Hamiltonian → only finite # of states with less than a given energy

  ↳ to preserve translational sym & SUSY, both bosons & fermions have periodic BC

  ↳ "For ordinary internal sym., could not use finite-vol limit to probe ∞ Vol breaking, since internal syms are always unbroken in finite volume" ← ?? Don't know why.
  However, for SUSY we can, since $\boxed{\text{SUSY unbroken} \Leftrightarrow E = 0}$
  (very straightforward)
  and $\lim_{V \to \infty} 0 = 0$

  ↳ if $E = 0$ for every finite V, $E = 0$ for ∞V
  → SUSY unbroken!

⇒ We will develop methods to prove that SUSY is unbroken (in finite $V \to \infty V$) for certain classes of theories.

↳ can't say when ~~SUSY~~ will occur for sure!

$\boxed{\text{CLAIM}: \quad n_B^{E=0} - n_F^{E=0} = 0 \Rightarrow \text{no spont } \cancel{\text{susy}}}$

and it can be reliably calculated in any convenient limit of parameters of the theory.

Proof

- Let $\mathcal{H}$ be the Hilbert space of our theory. If there is a zero-E state, susy is unbroken. → Restrict attention to $P=0$ subspace of $\mathcal{H}$

↳ SUSY algebra simplifies:

$$Q_i^2 = H$$
$$\{Q_i, Q_j\} = 0 \quad i \neq j$$

← Witten Notation: I think he writes
$Q_1 \to Q_1$
$Q_1^+ \to Q_2$
$Q_2 \to Q_3$
$Q_2^+ \to Q_4$
$Q_1 Q_1^+ \to$ "Q_1^2" etc

$i = 1, 2, \ldots, K$
$K = 4$ for $N=1$ 4D SUSY

↳ restrict attention to a single Q!

- In finite volume individual particle ill-defined, but can still define bosonic & fermionic states in theory.

$$(-1)^F |b\rangle = |b\rangle \quad , \quad (-1)^F |f\rangle = -|f\rangle$$

where $(-1)^F = e^{2\pi i J_z} = \left(e^{\frac{\pi}{2} i J_z}\right)^4$

well-defined sym. op in finite volume: quarter rotation

Then Q maps fermionic states to bosonic states & vice versa.

- States of nonzero E are paired by Q!!

$\forall |b\rangle$ is any bosonic state w/ energy E, define $|f\rangle = \frac{1}{\sqrt{E}} Q|b\rangle$

$\rightarrow \quad Q|b\rangle = \sqrt{E}\,|f\rangle , \qquad Q|f\rangle = \sqrt{E}\,|b\rangle$

$\Rightarrow$ $\boxed{\text{nonzero-}E \text{ states arrange themselves into super-multiplets!!}}$

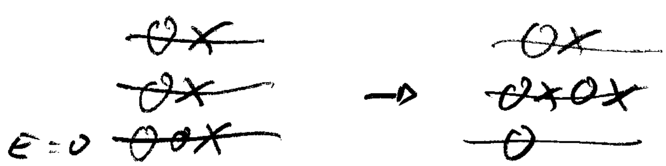
Not so for zero-E states

$\forall\ |b\rangle$ or $|f\rangle$ have zero E, $\quad Q|b\rangle = 0, \quad Q|f\rangle = 0.$

$\rightarrow$ $\boxed{\text{Arbitrary numbers } n_B^{E=0},\ n_F^{E=0} \text{ of zero } E \text{ states!}}$

- As we continuously change parameters of the theory (masses, couplings), $n_B^{E=0} - n_F^{E=0}$ does not change!!

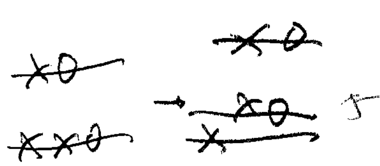
(e.g.



$n_B^{E=0} - n_F^{E=0} = 1$)

$\Rightarrow$ Can choose any convenient corner of parameter space to calculate $n_B^{E=0} - n_F^{E=0}$

↳ This makes it possible to calculate in almost any theory!



↳ even approximate calcs are OK: any error that misidentifies zero-E $|b\rangle$'s does the same for $|f\rangle$'s!!

NB: what if $n_B^{E=0} - n_F^{E=0} = 0$?

↳ A) $n_B^{E=0} = n_F^{E=0} = 0$ & ~~SUSY~~

or B) $n_B^{E=0} = n_F^{E=0} \neq 0$ & SUSY

A ⇒ Goldstino ⇒ massless fermion

B ⇒ $n_F^{E=0} \neq 0$ ⇒ massless fermions in ∞V limit

not entirely rigorous

So if $n_F^{E=0} - n_B^{E=0} = 0$, it seems reasonable that the ∞V thy has a massless fermion.

Definition: $\boxed{\mathrm{Tr}(-1)^F = n_B^{E=0} - n_F^{E=0}}$

↳ Comment: $\mathrm{Tr}(-1)^F$ is the INDEX of an operator.

↳ $\psi = \left(\frac{B}{F}\right) \Rightarrow Q = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}$ divide Hilb $\mathcal{H}_B$, $\mathcal{H}_F$

zero-E bosonic states satisfy $M^*\psi = 0$

— fermionic — — $M\psi = 0$

$$\Rightarrow \mathrm{Tr}(-1)^F = (\#\text{ solns to } M\psi = 0) - (\#\text{ solns to } M^*\psi = 0)$$

↳ Def'n of an index of op M!

⇒ proves $(-1)^F$ is invariant under small deformations/s of the theory!!

Important Subtleties regarding use of $Tr(-1)^F$

1) **UV divergences**: not important here, $Tr(-1)^F$ is an IR-sensitive quantity ✓

2) **Asymptotic Behavior of P.E. for large field strengths:**

↳ A perturbation that changes this behavior can permit new low-E states to "move in from ∞", changing $Tr(-1)^F$ discontinuously.

↳ eg $V = (m\phi - g\phi^2)^2$

$g = 0$: $V \sim \phi^2$ for large ϕ, low-energy states at $\phi = 0$

$g \neq 0$: $V \sim \phi^4$ for large ϕ, low-E states at $\phi = 0$ and $\phi \sim \frac{m}{g}$ ← NEW

} different $Tr(-1)^F$

⇒ $Tr(-1)^F$ is independent of numerical values of parameters in Hamiltonian as long as they are non-zero!

↳ Setting a coupling to zero or introducing a new one can change $Tr(-1)^F$!

(if it changes the asymptotic behavior of V)

3. Conjugation

(Note: only give summary of results. This section is very technical but not very useful.)

- Work w/ 2 supercharges. Define $Q_\pm = \sqrt{\tfrac{1}{2}}\,(Q_1 \pm i Q_2)$
- Consider <u>parameter changes</u> which can be brought about by the substitution

$$Q_+ \to \tilde{Q}_+ = M^{-1} Q_+ M, \quad Q_- \to \tilde{Q}_- = M^{+} Q_- M^{+\,-1}, \quad H \to \tilde{H} = \tilde{Q}_+\tilde{Q}_- + \tilde{Q}_-\tilde{Q}_+$$

↳ $\boxed{\text{"parameter changes that can be brought about by conjugation"}}$

↳ leaves $n_B^{E=0} + n_F^{E=0}$ const

$\Rightarrow n_B^{E=0},\ n_F^{E=0}$ seperately conserved!!!

- What kind of parameter changes can be brought about in this way?
  - superpotential parameters: YES
  - abelian gauge couplings: YES
  - non-abelian gauge couplings: YES, <u>IF</u> theory is Θ-indep.
  - Θ angles: <u>NO</u>
  - Fayet-Iliopoulos D-terms: <u>NO</u>

- $\boxed{\text{NOT VERY USEFUL IN PRACTISE, because } n_B^{E=0} + n_F^{E=0} \text{ is hard to calculate even at weak coupling.}}$

4. Analyticity

Consider first SUSY QM:

- finite # dof

↳ energy eigenvalues are analytical fns of Hamiltonian parameters.

↳ caveat; analogous to sec 2: deformations which change asymptotic behaviour of Hamiltonian can cause analyticity to break down

↳ GROUND STATE ENERGY IS ANALYTICAL FN OF PARAMETERS

⇒ if we know $E_0 = 0$ ($E_0 \neq 0$) in finite volume region of parameter space, SUSY unbroken (broken) everywhere in param space except ⊛ in lower-dim subspaces where analyticity might break down.

Move on to SUSY QFT

- Assume theory has UV cutoff Λ. Take $V, \Lambda < \infty$ ⇒ finite # dof

⇒ analyticity holds as above! (NOT TRUE in $V, \Lambda \to \infty$ LIMIT!!)

↳ 1) suppose $\forall\ \Lambda, V < \infty$, $\exists$ nonzero region of param space where SUSY UNBROKEN

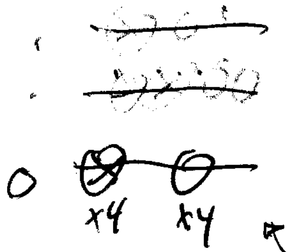
⇒ since $\lim_{V,\Lambda\to\infty} 0 = 0$, SUSY UNBROKEN IN $\Lambda, V \to \infty$ LIMIT $\forall$ param choices (except caveat ⊛)

2) if — " — $\exists$ — " — where SUSY is BROKEN, cannot say anything about $\Lambda, V \to \infty$ limit:

$\lim_{V,\Lambda\to\infty} (\neq 0)$ could be zero. → SUSY could be restored as $\Lambda, V \to \infty$!

Example of Application:

Suppose we find following result for finite V, Λ in weak coupling limit:



$\times$ = spin $\frac{1}{2}$

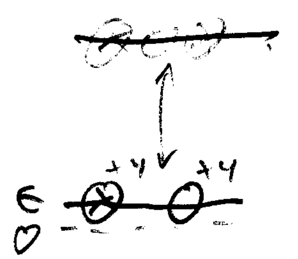
$\mathcal{O}$ = 0

$\otimes$ = $\frac{3}{2}$

(Note $Tr(-1)^F = 0$, so this doesn't help...)

do these zero-E states in weak-coupling limit/approx REALLY have zero E for $g \ll 1$ but $g > 0$?

↳ Well, certainly for $g \lll 1$, their energy, if nonzero, is $\ll$ the other states in the theory:



So Q's can only switch these states amongst each other. But Q can't change spin $\frac{3}{2}$ to 0!

spin $\frac{1}{2}$

→ Q must annihilate these states

→ $n_F^{E=0}, n_B^{E=0} \neq 0$ → SUSY unbroken for range of params for $\Lambda, V < \infty$

→ by analyticity, SUSY unbroken for $\Lambda, V < \infty$ on entire param space → for $\Lambda, V \to \infty$ too!! □

5. Simple Applications

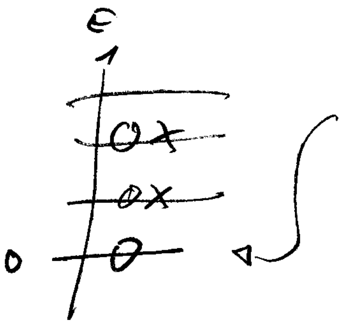
Wess Zumino Model

- $W = \frac{1}{3} g \phi^3 - \frac{m^2}{4g}\phi \longrightarrow V = g^2 \left|\phi^2 - \frac{m^2}{4g^2}\right|^2$

  $\mathcal{L}_{Yuk} = g \phi \psi\psi + hc$

- Evaluate $Tr(-1)^F$ in finite volume for $m \neq 0$

  ↳ $\langle\phi\rangle = \pm m/2g$ and $m_\phi = m_\psi = m(1 + O(g^2))$

$\Rightarrow$ 

zero-E state is the bosonic vacuum $\langle\phi\rangle = \pm m/2g$.

Other states are obtained by adding ϕ, ψ excitations to the vacuum, which are massive $\rightarrow$ increase energy.

$\Rightarrow Tr(-1)^F = 2 \rightarrow$ SUSY unbroken!

- What if $m \simeq 0$?

  ↳ Does not change behavior of V as $\phi \to \infty$, $V \sim \phi^4$

  $\rightarrow$ SAME CONCLUSION!

- We already knew that the WZ model does not break SUSY spontaneously for small g. What have we gained by using $Tr(-1)^F$?

  ↳ one might think that there is some critical value of g where $m_\psi = 0$ & ψ becomes a Goldstino of ~~SUSY~~.

  CANNOT HAPPEN DUE TO $Tr(-1)^F$ INDEPENDENCE OF g!

  ↳ in $m=0$ case, maybe some small NP effect gives $\langle v \rangle \neq 0$?

  NOPE! $Tr(-1)^F$!

Other Models

- Same argument as above applies to ANY model w/ SUSY unbroken @ tree level **and** all particles massive. → SUSY UNBROKEN!

  ↳ if particles are massless, argument still applies as long as I could make particle massive by changing params of the theory w/o changing asymptotic behavior of potential. (see ⊛ on prev. page)

- $\boxed{\text{ARGUMENT BREAKS DOWN WHEN THEORY HAS MASSLESS PARTICLES THAT CAN'T BE MADE MASSIVE! → eg GAUGE THEORIES (w/o SSB)}}$

6. Abelian Gauge Theories

Model Definition

Concentrate on $U(1)$ gauge theory with VECTOR-LIKE MATTER, so that mass terms are allowed.

↳ can assume these bare masses to be present when calculating $Tr(-1)^F$ (is indep of bare masses, so can set them to zero later)

↳ CHIRAL matter (no possible mass terms) would be much more complicated!

Useful Generalization of $Tr(-1)^F$

Let X be some conserved charge in the theory that commutes with the SUSY algebra. $[X, Q] = 0$

→ we can restrict $Tr(-1)^F$ to a subspace of $\mathcal{H}$ with X-eigenvalue λ: $Tr(-1)^F P_\lambda$ ← projection op. (or equiv. $Tr(-1)^F f(X)$)

→ $Tr(-1)^F P_\lambda$ can be used just like regular $Tr(-1)^F$ to determine whether SUSY is not broken, but more powerful because we can generate many operators using this method and use them independently. even if $Tr(-1)^F = 0$ one of $Tr(-1)^F P_\lambda$ might not be!

→ It's OK if X is spontaneously broken in ∞V limit

As long as X is well defined and commutes with Q,

$\boxed{\mathrm{Tr}(-1)^F f(X)}$ can be used to get info on ~~SUSY~~.

⟹ In vector-like QED, we use $X = C$ (charge conjugation op).

↳ NB: non-minimal couplings could violate C. HOWEVER, if we can remove those violations using parameter changes generated by "conjugation" (see Sec 3), then we can still use C because these changes would leave $n_{F,B}^{E=0}$ independently invariant.

↳ Under C, ψ_A & A_μ change sign

↳ $C^2 = 1$ ⟹ two independent variants:

$\mathrm{Tr}(-1)^F$ ← will find: zero

$\mathrm{Tr}(-1)^F C$ ← NONZERO ⟹ no ~~SUSY~~!

To Calculate $Tr(-1)^F$: Neat trick involving D-terms

- say there are no non-minimal interactions
- C_i = charged scalars $\rightarrow$ $V(C_i) = \sum m_i^2 |C_i|^2 + \underbrace{\left(\sum e_i |C_i|^2\right)^2}_{\text{D-terms}}$

  $\langle C_i \rangle = 0$, no ~~SUSY~~ @ tree BUT WHAT ABOUT QM THY?

- Add FI-D-term:

$$V = \sum m_i |C_i|^2 + \left(\sum e_i |C_i|^2 - d^2\right)$$

For $d \neq 0$, have ~~SUSY~~ @ tree. as long as $m_i \neq 0$.

$\hookrightarrow$ $Tr(-1)^F$ indep of D-term, so $\boxed{Tr(-1)^F = 0}$ for $d = 0$

$\rightarrow$ NO INFORMATION! $\Rightarrow$ Need $tr(-1)^F$ generalizations!

- Can we use this trick to calculate $Tr(-1)^F C$? <u>NO</u>

$\hookrightarrow$ $\forall\, C_i$ w/ charge e_i, $\exists\, C_j$ w/ charge $e_j = -e_i$ $\Rightarrow C: C_i \leftrightarrow C_j$

$\Rightarrow e_i \leftrightarrow -e_i$ $\Rightarrow$ EFFECTIVELY $d^2 \rightarrow -d^2$ in the scalar potential

$\Rightarrow$ D-term <u>not</u> invariant under charge conjugation

$\Rightarrow$ cannot use C to generate variants of $Tr(-1)^F$ for $d \neq 0$, and we cannot deform $d \rightarrow 0$ via "conjugations"

$\Rightarrow$ $\boxed{\text{D-terms do not help calculating } Tr(-1)^F C!}$

General Approach to Calculating $\mathrm{Tr}(-1)^F$, $\mathrm{Tr}(-1)^F C$

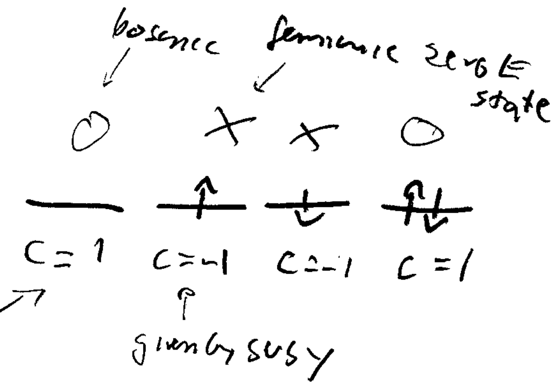
- Key: if all charged fields are massive $\underline{(m_i \neq 0)}$, they do not contribute to $\mathrm{Tr}(-1)^F f(x)$

  $\Rightarrow$ evaluate everything in free SQED,

  $\boxed{\text{only care about massless } A_\mu, \Psi_A.}$

- zero-momentum mode of massless fields can contribute to $\mathrm{Tr}(-1)^F$

FERMION CONTRIBUTION (Ψ_A):



↳ 4 zero-mom fermion states fit in Box:

can set C of — to be 1 wlog.

$\Rightarrow \ \mathrm{Tr}(-1)^F = 0 \qquad \boxed{\mathrm{Tr}(-1)^F C = 4}$

GAUGE FIELD CONTRIBUTION (A_μ)

$\boxed{\text{will show does not contribute to } \mathrm{Tr}(-1)^F, \ \mathrm{Tr}(-1)^F C}$

First, a note on gauging the zero momentum mode away:

↳ if $V = \infty$, $A_\mu \to A_\mu + \partial_\mu \epsilon$ with $\epsilon = c_\mu x^\mu$ gauges the zero momentum mode $A_\mu = \text{const}$ away.

↳ if $V < \infty$, $A_\mu = (A_0, \vec{A})$ can be put in the form $(0, \vec{A})$ via the above gauge transformation, since time is not periodic.

What about $\vec{A}$?

$\epsilon = c_\mu x^\mu$

- Without charged fields:

  ↳ $A_i \to A_i + \partial_i \epsilon$ preserves periodicity of A_i

  ⇒ eliminate zero-mom mode, contributes nothing to $\mathrm{Tr}(-1)^F$, $\mathrm{Tr}(-1)^F C$!

- With charged fields:

  ↳ charged fields transform $\psi_{e_j} \to e^{i e_j \vec{c}\cdot\vec{x}} \psi_{e_j}$

  ⇒ assuming all charges e_j are integer multiples of some basic charge e, requiring all the ψ_{e_j} to <u>remain periodic</u>, restricts the gauge freedom to $\epsilon = c_\mu x^\mu$ with $\boxed{c_i = \frac{2\pi}{eL} n_i}$ L = length of box

  → can NOT simply eliminate zero mode of gauge field.

↳ define zero-mom mode of gauge field as

$$h_i = \frac{1}{V}\int d^3x\, A_i$$

gauge transformations can shift it by $h_i \to h_i + \frac{2\pi}{eL} n_i$

⇒ h_i are spatially periodic with period $\frac{2\pi}{eL}$

⇒ quanta of h_i have nonzero Energy

⇒ zero-mom mode of gauge field does NOT CONTRIBUTE to $\mathrm{Tr}(-1)^F$

Remarks

1) This whole $Tr(-1)^F$ business assumes H has discrete spectrum of states in finite-volume case.

↳ NOT NECESSARILY TRUE WHEN THERE ARE MASSLESS PARTICLES because the zero-momentum (zero mass) modes may have a cts spectrum

↳ what saves us? gauge invariance restricts us to the subset of Hilbert space which is invariant under $A_i \to A_i + \frac{2\pi}{eL} h_i$

→ periodic → discrete → $Tr(-1)^F$ OK!

2) We said nothing about dynamical breaking (or absence thereof) of C, or that ground states in $V \to \infty$ limit are C-eigenstates.

3) Like for WZ model, we could've deduced lack of ~~SUSY~~ for weak coupling using different methods easily.

↳ $Tr(-1)^F$, $Tr(-1)^F C$ works at strong coupling too!!!
(and even after removing bare mass terms of the charged fields!)

Summary

conserved charge

- If $Tr(-1)^F$ or $Tr(-1)^F f(x)$ is NONZERO in ANY LIMIT of theory, NO DYNAMICAL ~~SUSY~~!

  ↳ Careful with setting couplings to zero or introducing new masses!

- When is $Tr(-1)^F$ or $Tr(-1)^F f(x)$ nonzero?

  ↳ any model w/ tree-level unbroken SUSY & all particles massive (or possibly massive)

  ↳ U(1) vector-like QED

Next Week

Josh will show that $tr(-1)^F \neq 0$ for vector-like simple non-abelian gauge theory.

Take-Home Message

no ~~SUSY~~:
- massive particles w/o ~~SUSY~~ @ tree
- vector-like gauge theories

maybe ~~SUSY~~:
- chiral gauge theories