Preprint typeset in JHEP style - PAPER VERSION

Massive Neutrinos in a Warped Extra Dimension

Philip Tanedo

Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Cambridge, CB3 0WA, UK
Email: pt291@cam.ac.uk

ABSTRACT: The Randall-Sundrum I (RS1) model of a warped extra dimension provides a natural candidate solution to the hierarchy problem between the Planck and weak scales. Coincidentally, the theoretical development of such 'braneworld' models in the late 1990s coincided with the experimental verification of nonzero neutrino masses. This presents another hierarchy problem: why are neutrino masses vanishingly small compared to the weak scale? To solve this, Grossman and Neubert have proposed a 'generalized see-saw mechanism' utilizing bulk right-handed neutrinos within the RS1 framework. Such a model is highly constrained by limits on lepton flavor violation, and Kitano has shown that some fine tuning is required to meet these bounds. In this essay I will present the RS1 model, the Grossman-Neubert extension, and lepton flavor violation from RS1 bulk neutrinos.

ESSAY 74, THE PHENOMENOLOGY OF EXTRA DIMENSIONS. Extra dimensional models provide an interesting playground for model building and investigating collider signatures. Candidates are invited to provide an overview of one of the following extra-dimensional models: ADD (Arkani-Hamed, Dimopoulos and Dvali), UED (Universal Extra Dimensions) or Randall-Sundrum I. The candidate should include a calculation of a matrix element squared for a collider signature of the model.

This essay is presented in partial fulfillment of the requirements for a Certificate of Advanced Study in Mathematics at the University of Cambridge.

Contents

1. Introduction: The Spirit of '98

After three decades under the hegemony of the Standard Model/Minimally Supersymmetric Standard Model, the end of the 1990s was marked by surprising new theoretical and experimental directions beyond the Standard Model. In 1998-'99, papers by Arkani-Hamed, Dimopoulos, and Dvali (ADD) [1] as well as Randall and Sundrum (RS1) [2] introduced modern[1] extra-dimensional 'braneworld' models that provided novel approaches to the hierarchy problem. The RS1 model, in particular, features a warped metric that generates a Planck-weak hierarchy with natural $\mathcal{O}(1)$ dimensionless parameters.

Just four months after the first of these braneworld papers, the Super-Kamiokande collaboration published atmospheric neutrino results indicating that neutrinos have non-zero, but very tiny, mass [5]. This was the first experimental result in particle physics that required a modification of the Standard Model Lagrangian. Just when theorists had made progress on the relation between the Planck and weak scales, the discovery of these vanishingly small neutrino masses introduced a new hierarchy problem.

The standard approach to generating this scale is through the see-saw mechanism by which massless left-handed neutrinos mix with a heavy right-handed neutrino to form eV-scale mass eigenstates [6]. However, with the timely development of braneworld models to address the original hierarchy problem, the natural step would be to look for inherently extra dimensional solutions to the neutrino mass question. The RS1 model presents unique challenges for neutrino model-building, since there is neither an intermediate energy scale available to see-saw neutrino masses nor an appreciable extra dimensional volume suppression as implemented in the ADD model [7].

Grossman and Neubert have presented a minimal extension to the RS1 model where the right-handed neutrino, which is a Standard Model gauge-singlet, is allowed to propagate in the bulk [8]. Suitable boundary conditions allow a zero mode that is localized on the hidden brane, utilizing the RS1 warp factor to generate the neutrino mass hierarchy. This generalizes the see-saw mechanism to warped extra dimensions. Kitano, Cheng, and Li have shown that such a model is constrained by lepton flavor violation and that experimental bounds force some fine-tuning of model's parameters [9, 10].

In this essay I review the RS1 model, its extension by Grossman and Neubert, and Kitano's calculation of lepton flavor violation from $\mu \to e\gamma$. In Section 2 I present the RS1 model and its solution to the hierarchy problem. In Section 3 I introduce the formalism of bulk fermions within a curved space. I then make use of this formalism in Section 4, where I discuss the Grossman-Neubert extension for neutrino masses. In Section 5 I show that such an extension generates lepton flavor violation effects that are constrained by experimental bounds; I have included recent experimental results published after Kitano's original analysis. My notation and conventions are summarized in Appendix A. Some calculations regarding the bulk fermion formalism that are not explicit in the Grossman-Neubert paper are worked out in Appendix B. The details of the calculation of the $\mu \to e\gamma$ amplitude in the Grossman-Neubert extension are presented in Appendix C.

2. The RS1 Model

We begin by describing the RS1 model and the mechanism by which it generates the Planck-weak hierarchy. A cartoon picture of the model is presented in Figure 1. Our observed space is actually one of two 3+1 dimensional branes at the endpoints of an interval in a fifth dimension. The bulk 4+1

[1] Here 'modern' is meant to distinguish these from the original braneworld scenarios developed independently in 1982 by Akama [3] and in 1983 by Rubakov and Shaposhnikov [4]. It wasn't until ADD and RS1, however, that braneworld models were connected to the hierarchy problem.

dimensional spacetime is warped in such a way that the induced metric on our brane is conformally scaled. The measure of this rescaling is the warp factor, which is shown as a red line. We shall see that this warping also has the effect of rescaling masses relative to the fundamental 5D Planck mass, thus generating the hierarchy. Before proceeding, some introductory remarks are in order regarding the nature of branes.

2.1 Welcome to braneworld

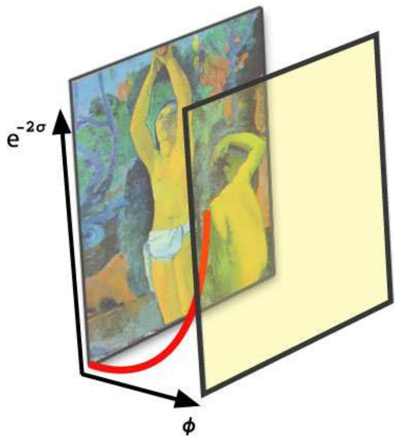

Figure 1: RS1 braneworld with the visible brane represented by Gauguin's "*Where Do We Come From? What Are We? Where Are We Going?*" (1897). The red line depicts the warp factor, with the ϕ axis running from $-\pi$ to 0.

As prompted by Gauguin's famous painting in Figure 1, the natural introductory questions to ask are, "*Where do* [branes] *come from? What are* [branes]*? Where are* [branes] *going?*" Following the typical response of physics students to Post-Impressionist art, the correct answer for low-energy model-builders is *who cares?*

We shall take Sundrum's model-building definition of a **brane** as a 3+1 dimensional surface in a higher dimensional space where standard model particles are confined to propagate [11]. These objects are solitonic membranes of characteristic width much smaller than the scale of the low-energy effective theory we seek. The existence of such objects can be motivated by a high energy theory. In string theory, for example, the endpoints of open strings can be constrained to fall on so-called *D*-branes. Randall-Sundrum models, in particular, are reminiscent of Horava-Witten domain walls in M-theory [12, 13]. But more generally, the point is that brane-like topological defects are allowed to exist in the low-energy effective theory of some high-energy theory and that the effective theory will be insensitive to the particular high-energy mechanism that generated the brane[2]. Thus low-energy theorists needn't worry about why or how branes exist, only whether or not realistic models can be built from them.

The relevant question, then, isn't where branes come from, but rather what they can offer as objects in an effective theory. The original Kaluza-Klein ansatz suffered from the precision data of particle accelerators that constrained the size of an extra dimension to be much smaller than 10^{-16} cm. The key insight by Akama, Rubakov, and Shaposhnikov was that this constraint only holds if Standard Model particles propagate in the extra dimension [3, 4]. If only gravity is allowed to propagate in the bulk, then the size of the extra dimension is more weakly constrained by experiments probing Newton's laws. Hence the braneworld scenario allows us to escape the glaring observation that everything we see appears to be 3+1 dimensional.

As Rattazzi explains in his Cargese lectures [14], the RS1 scenario takes advantage of the braneworld scenario in another way. The **hierarchy problem** can be posed as the question of why the characteristic mass of the weak scale is so much smaller than that of the Planck scale. A particle's mass (energy) is identified with its frequency as a quantum field. We already know of a classical mechanism where frequencies are made small: gravitational redshift. From general relativity we know that the curvature of space redshifts photons near a massive object. We shall see that the RS1 scenario will

[2] In fact, the general nonrenormalizability of higher dimensional theories *requires* such models to be effective theories.

use the curvature of the five-dimensional bulk to analogously 'redshift' the 5D Planck scale into the 4D weak scale.

2.2 Defining the interval: Orbifolding

The extra dimensional interval in the RS1 scenario is formally the **orbifold** $S^1/\mathbb{Z}_2$. Orbifolding is the process by which a manifold is 'modded out' by a discrete symmetry [15, 16]. We turn the circle S^1, parameterized by an angular variable ϕ, into an interval by identifying points $\phi \in [-\pi, \pi]$ via $\phi = -\phi$. This is represented in Figure 2. Orbifolding this way imposes a parity symmetry $\mathcal{L}(x, \phi) = \mathcal{L}(x, -\phi)$ which will play an important role in removing unwanted degrees of freedom in Section 4. The variable ϕ still ranges from $-\pi$ to π but space is completely specified by its values from 0 to π. The resulting orbifold isn't a manifold, but rather a 'manifold with boundary.'

2.3 The wind up: The RS1 action

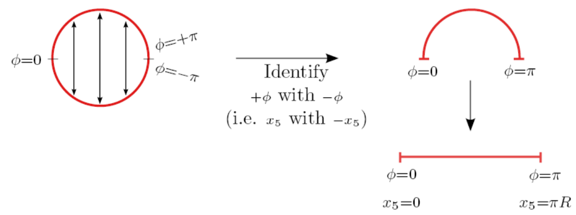

Figure 2: The RS1 orbifold $S^1/\mathbb{Z}_2$ is constructed by identifying points on the circle, figure from [16].

The RS1 set up is given by a five-dimensional spacetime where the fifth dimension is the orbifold $S^1/\mathbb{Z}_2$, at the endpoints of which are two 3+1 dimensional branes labeled the **visible** and **hidden** branes. Standard Model fields are localized the visible brane and only gravity is allowed to propagate in the bulk. The bulk space is allowed to be curved (we shall see that it is AdS_5), but we will want the visible brane to be flat with respect to the induced 3+1 dimensional metric. In order to impose this we allow the branes to contain uniform 3+1-dimensional energy densities Λ_{vis} and Λ_{hid} which we may interpret as **brane tensions** or **brane cosmological constants**. Given these assumptions, the action can be written in terms of a bulk gravitational action S_{G} with brane-localized actions S_{vis} and S_{hid}.

$$S = S_{\text{G}} + S_{\text{vis}} + S_{\text{hid}}. \tag{2.1}$$

As there are no bulk fields, S_{G} is just the 5D Einstein-Hilbert action. The visible brane action S_{vis} and the hidden brane action S_{hid} are allowed to have localized fields. Thus we can write the above terms as

$$S_{\text{G}} = \int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{G} \left\{ M^3 R - \Lambda \right\} \tag{2.2}$$

$$S_{\text{vis}} = \int d^4x \, \sqrt{g_{\text{vis}}} \left\{ \mathcal{L}_{\text{vis}} - \Lambda_{\text{vis}} \right\} \big|_{\phi=\pi} \tag{2.3}$$

$$S_{\text{hid}} = \int d^4x \, \sqrt{g_{\text{hid}}} \left\{ \mathcal{L}_{\text{hid}} - \Lambda_{\text{hid}} \right\} \big|_{\phi=0} . \tag{2.4}$$

Here g, g_{vis}, and g_{hid} are the positive determinants of the 5D and brane metrics. The fundamental scale of the theory is the 5D Planck mass M, R is the 5D Ricci scalar, and Λ is the bulk cosmological constant. $\mathcal{L}_{\text{vis}}$ and $\mathcal{L}_{\text{hid}}$ are brane-localized Lagrangians with $\mathcal{L}_{\text{vis}}$ assumed to include the Standard Model. The physics of $\mathcal{L}_{\text{hid}}$ are irrelevant to our effective theory[3].

[3] In non-minimal models $\mathcal{L}_{\text{hid}}$ can play an important role in accommodating a SUSY-breaking sector.

Let's begin with an ansatz for the form of the metric:

$$ds^2 = e^{-2\sigma(\phi)}\eta_{\mu\nu}dx^\mu dx^\nu - r^2 d\phi^2 \tag{2.5}$$

Here r is the compactification radius, a parameter of our theory. The function $\sigma(\phi)$ must be determined and is known as the **warp factor.** One can already see that it will be responsible for the generation of a large mass hierarchies by 'redshifting' 4D proper distance depending on the brane position along the orbifold interval.

Armed with this ansatz, we would like to determine the classical ground state of the theory using Einstein's equation, which relates the Einstein tensor $G_{MN} = R_{MN} - 1/2Rg_{MN}$ to the stress-energy tensor T_{MN},

$$G_{MN} = \kappa^2 T_{MN} = \frac{g_{MN}}{2M^3}\Lambda + \frac{g_{MN}}{2M^3}\left(\Lambda_{\text{hid}}\delta(\phi) + \Lambda_{\text{vis}}\delta(\phi-\pi)\right)|_{M,N\neq 5}. \tag{2.6}$$

The right-hand side tells us that the energy momentum tensor is given by the contribution from the bulk cosmological constant and brane tensions, where brane tension terms are localized by δ-functions and are zero if the index M or N runs over the fifth dimension. In Section 4 we will add a neutrino to the bulk action, but such a term will not be relevant for determining the ground state of this theory. κ^2 is a constant proportional to M^{-3} and is related to the higher dimensional Newton's constant. We shall choose our normalization[4] to be $\kappa^2 = \frac{1}{4M^3}$.

Our strategy will be to apply Einstein's equation (2.6) to our metric ansatz (2.5) to constrain $\sigma(\phi)$ and the brane cosmological constant terms.

2.4 The pitch: RS1 background solution

One may follow the original Randall-Sundrum paper [2] and immediately solve Einstein's equation, but this is unsatisfying as it requires an awkward combination of brute force and calculational finesse. Instead, in this section we shall follow the slick and pedagogical approach in Csaba Csaki's 2004 TASI lectures [18].

Let us begin by changing to a conformally flat coordinate system by introducing a new coordinate z such that

$$ds^2 = e^{-A(z)}\left(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2\right). \tag{2.7}$$

This allows us to invoke a nice relation between the Einstein tensors of conformally related d-dimensional metrics $g_{MN} = e^{-A(x)}\tilde{g}_{MN}$, [19]:

$$G_{MN} = \tilde{G}_{MN} + \frac{d-2}{2}\left[\frac{1}{2}\tilde{\nabla}_M A\tilde{\nabla}_N A + \tilde{\nabla}_M\tilde{\nabla}_N A - \tilde{G}_{MN}\left(\tilde{\nabla}_K\tilde{\nabla}^K A - \frac{d-3}{4}\tilde{\nabla}_K A\tilde{\nabla}^K A\right)\right].$$

In our case, $\tilde{g}_{MN} = \eta_{MN}$, so the covariant derivatives become partial derivatives, $\tilde{\nabla}_M \to \partial_M$. One can then read off the 55 and $\mu\nu$ components of the Einstein tensor for our metric (2.7),

$$G_{55} = \frac{3}{2}A'^2 \tag{2.8}$$

$$G_{\mu\nu} = \frac{3}{2}\eta_{\mu\nu}\left(A'' - \frac{1}{2}A'^2\right). \tag{2.9}$$

[4]There is some arbitrariness regarding the constant of proportionality between the Newton's constant and κ^2. Those who are particularly perturbed by this can peruse the discussion by Robinson [17].

We will now proceed in two steps and solve Einstein's equation (2.6) separately for the 55 and $\mu\nu$ components. The first step will determine $\sigma(\phi)$ while the second will constrain $\Lambda_{\rm vis}$ and $\Lambda_{\rm hid}$.

The 55 component is independent of the brane tension terms and gives

$$-\frac{3}{2}A'^2 = -\frac{1}{4M^3}\Lambda e^{-A(z)}, \tag{2.10}$$

from which we can write

$$A' = e^{-A(z)/2}\sqrt{-\frac{\Lambda}{6M^3}}. \tag{2.11}$$

The sign inside the square root imposes a negative cosmological constant $\Lambda < 0$, and hence the bulk space is five dimensional anti-de Sitter (AdS_5). We can solve this equation using another trick. Define $f \equiv e^{-A/2}$ and plug into equation (2.11) to get

$$-\frac{f'}{f^2} = \frac{1}{2}\sqrt{-\frac{\Lambda}{6M^3}}. \tag{2.12}$$

The general solution of this differential equation is

$$e^{-A(z)} = \frac{1}{(kz+C)^2}, \tag{2.13}$$

$$k \equiv -\frac{\Lambda}{12M^3}. \tag{2.14}$$

The constant is fixed by imposing $e^{-A(0)} = 1$, and hence our conformally flat metric takes the form:

$$ds^2 = \frac{1}{(k|z|+1)^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2\right). \tag{2.15}$$

We have made the critical replacement $z \to |z|$ to maintain the $S^1/\mathbb{Z}_2$ orbifold symmetry $\phi \to -\phi$ (i.e. $z \to -z$). W now invert the definition of the conformal coordinate z and the conformal factor $A(z)$ in equation (2.7). By integrating $r^2\, d\phi^2 = (k|z|+1)^{-1}dz^2$ and imposing $e^{-2\sigma(\phi)} = (k|z|+1)^{-1}$, we find that $\sigma(\phi) = kr|\phi|$, as depicted heuristically in Figure 1. The metric, in (x,ϕ) coordinates, is thus

$$ds^2 = e^{-2kr|\phi|}\eta_{\mu\nu}dx^\mu dx^\nu - r^2 d\phi^2. \tag{2.16}$$

As promised, the 55 component of Einstein's equation has fixed the warp factor. We now proceed to the second step, solving the $\mu\nu$ components (2.9). From the form of the $\mu\nu$ equation one can already see the necessity of the brane tensions (2.3-2.4). Because A depends on the modulus of z, the second derivative terms in (2.9) will generate δ functions at the orbifold boundaries $z = 0, z_1$:

$$A'' = -\frac{2k^2}{(k|z|+1)^2} + \frac{4k}{k|z|+1}\left(\delta(z) - \delta(z-z_1)\right). \tag{2.17}$$

These δ-functions must be compensated by constant energy densities localized on the branes, i.e. brane tensions. Physically, these brane tensions compensate the 5D bulk cosmological constant so that the induced 4D brane metric is flat. Inserting equations (2.11) and (2.17) into (2.9),we have

$$G_{\mu\nu} = -\frac{3}{2}\eta_{\mu\nu}\left[\frac{4k^2}{(k|z|+1)^2} - \frac{4k(\delta(z)-\delta(z-z_1)}{k|z|+1}\right]. \tag{2.18}$$

Using the definition of k in (2.14), we see that the first term in $G_{\mu\nu}$ above cancels the bulk cosmological constant in the energy momentum tensor (2.6). The remaining δ function terms must correspond to the brane tensions such that

$$-\frac{3}{2}\eta_{\mu\nu}\left[-\frac{4k(\delta(z)-\delta(z-z_1))}{k|z|+1}\right] = \frac{\eta_{\mu\nu}}{4M^3}\left[\frac{\Lambda_{\text{hid}}\delta(z)-\Lambda_{\text{vis}}\delta(z-z_1))}{k|z|+1}\right]. \tag{2.19}$$

Finally, we discover that the brane tensions must be given by

$$\Lambda_{\text{vis}} = -\Lambda_{\text{hid}} = \sqrt{-\frac{\Lambda}{24M^3}}. \tag{2.20}$$

Let us briefly review what we've done. In searching for a background classical ground state of our AdS_5 system with an $S^1/\mathbb{Z}_2$ orbifold, we used the 55 component of the Einstein equation to determine the form of the warp factor $\sigma(\phi)$ and then used the $\mu\nu$ component to determine the brane tensions. Let's now connect this to the hierarchy problem.

2.5 A home run: Generating the hierarchy

In order to explain the hierarchy we will first need to understand the low-energy 4D theory that is generated by the RS1 scenario. We are especially interested in writing the 4D Planck mass M_{Pl} and the Standard Model masses in terms of the remaining unconstrained 5D parameters M, k (or Λ), and r. We shall follow the approach in the original RS1 paper [2].

First let us derive M_{Pl}. We assume that the radius r is fixed at some constant value. 4D graviton excitations $h_{\mu\nu}(x)$ can be inserted into the metric 'on top' of the flat 4D metric as follows,

$$ds^2 = e^{-2kr|\phi|}(\eta_{\mu\nu} + h_{\mu\nu}(x))dx^\mu dx^\nu - r^2 d\phi^2. \tag{2.21}$$

Taking the curvature of the $h_{\mu\nu}(x)$ perturbation into account, we get an additional contribution to the bulk gravitational action (2.2),

$$\Delta S_g = M^3 \int d^4x \int_{-\pi}^{\pi} r\, d\phi\, e^{-4kr|\phi|}\sqrt{\tilde{g}}\, e^{2kr|\phi|}\tilde{R}, \tag{2.22}$$

where $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$ and $\tilde{R}$ is the 4D Ricci tensor formed by $\tilde{g}_{\mu\nu}$. By performing the ϕ integral we get a contribution to the 4D effective action whose coefficient is the 4D effective Planck mass M_{Pl}. Explicitly, we have

$$M_{\text{Pl}}^2 = M^3 r \int_{-\pi}^{\pi} d\phi\, e^{-2kr|\phi|} = \frac{M^3}{k}\left(1 - e^{-2kr\pi}\right). \tag{2.23}$$

The key feature in the above equation is that, unlike the ADD scenario, it is rather insensitive to the size of the extra dimension r. Further, if we let the 5D parameters take natural values near the fundamental Planck scale so that $k \sim M$, then the 4D Planck mass is the same order as the 5D Planck mass, $M_{\text{Pl}} \sim M$.

Now consider the generation of the Standard Model masses. The mass scale for the $SU(2)\times U(1)$ weak sector is set by the Higgs vacuum expectation value. In the RS1 model, the Standard Model Lagrangian is part of $\mathcal{L}_{\text{vis}}$ in equation (2.3). The presence of the warp factor in $\sqrt{g_{\text{vis}}}$ and implicitly in the contraction of vector indices will force us to rescale our fields to maintain canonical normalization.

This rescaling will be the source of the exponential suppression of the weak scale relative to the Planck scale. Consider the Higgs sector on the visible brane,

$$S_{\rm H} = \int d^4x\sqrt{g_{\rm vis}}\left[g_{\rm vis}^{\mu\nu}D_\mu H D_\nu H - \lambda(|H|^2 - v_0^2)^2\right]|_{\phi=\pi} \tag{2.24}$$

$$= \int d^4x\; e^{-4kr\pi}\sqrt{\tilde{g}}\left[e^{2kr\pi}\;\tilde{g}^{\mu\nu}D_\mu H D_\nu H - \lambda(|H|^2 - v_0^2)^2\right] \tag{2.25}$$

Now watch carefully, this is the magical part. In order to work in an effective 4D low-energy theory, we need to canonically normalize our Higgs field $H \to e^{kr\pi}H$ and so we write this above line as

$$S_{\rm H} = \int d^4x\;\sqrt{\tilde{g}}\left[\tilde{g}^{\mu\nu}D_\mu H D_\nu H - \lambda(|H|^2 - e^{-2kr\pi}v_0^2)^2\right]. \tag{2.26}$$

This tells us that the effective Higgs action takes its usual 4D form with the vacuum expectation value given by $v = e^{-kr\pi}v_0$. Since masses are generated by the Yukawa terms after electroweak symmetry breaking, we see that any mass term m_0 is also rescaled by the same factor,

$$m = e^{-kr\pi}m_0. \tag{2.27}$$

This wonderful result states that dimensionful quantities on the brane are warped to the weak scale while dimensionless parameters are left unchanged.

Unlike the 4D effective Planck mass $M_{\rm Pl} \sim M$, the masses of the electroweak Standard Model particles are exponentially sensitive to the product kr. To avoid fine-tuning and a hierarchy problem, we expect the fundamental dimensionful parameters M, k (or Λ), v_0, and $1/r$ take natural values on the order of the 5D Planck scale, M. We see from (2.27) that the 15 orders of magnitude between the Planck and weak scale can be successfully generated with natural values of $kr \approx 30$. We've thus eliminated the need for excessive fine-tuning and have provided a natural explanation of the Planck-weak hierarchy.

It is notable that this is fundamentally different from the 'solution' of the hierarchy problem proposed in the ADD scenario since we have actually explained how the hierarchy is generated from parameters with natural values. In the ADD model, on the other hand, one only swaps fine-tuning in the mass parameters for fine-tuning in the radius of compactification.

2.6 Dust under the rug

Before we become overzealous, let us note, for completeness, a few topics that we have swept under the rug. As these lie beyond the scope of this essay, we will necessarily be brief but will provide references for further discussion. First of all, we should note that although we have removed fine-tuning from the Planck-weak hierarchy, the RS1 model contains is finely tuned in the values of the brane tensions in equation (2.20) that are required for a static background. Physically, one of these tunings permits the flatness of the 4D metric.

The other tuning is related to our assumption that the radius r is fixed at a reasonable value [18]. In a more complete treatment, we would let $r = r(x, \phi)$ be a dynamical degree of freedom called the **radion**. It is associated with the 4D scalar component arising from the decomposition of the 5D metric [16]. Because the radion has no potential in our theory, it is a massless particle whose phenomenology would violate the equivalence principle and Newton's law. Thus there must be a mechanism to stabilize the radion moduli and dynamically fix the radius of our extra dimension to a natural value. A standard solution in the RS1 model is given by the Goldberger-Wise mechanism in which radion kinetic and potential terms conspire against one another to create a radion potential with a desirable vacuum [20]. Reviews of this mechanism are available in [14] and [18].

A few words are also in order about some 'fancy techniques.' The RS1 model's use of an AdS_5 bulk space makes it a natural candidate for formal theorists to think of holography and AdS/CFT connections to strongly coupled 4D systems. Raman Sundrum has advocated the AdS/CFT correspondence as a tool for warped space model builders analogous to dimensional analysis: it's not strictly necessary, but it a useful check to avoid errors [16]. Useful references for this are [21–23].

Finally, we have not mentioned the collider phenomenology of the Kaluza-Klein (KK) modes of the RS1 graviton. Unlike the ADD scenario, the graviton couplings with matter are on the order of the weak scale, not the Planck scale. Thus, instead of a near continuum of KK modes, the RS1 scenario offers a small number of KK excitations that can be individually detected. Above the TeV scale the gravitons become strongly coupled and one would expect RS1 to break down as an effective theory and a more fundamental theory of quantum gravity to become relevant [2]. Useful reviews are [14, 16, 18, 24].

3. Fermions in Extra Dimensions

We now take a short detour to explain the formalism of fermions in curved extra dimensions, which we shall make use of in Section 4 when we extend RS1 to incorporate bulk neutrinos. Fermions are described by the spin-1/2 representation of the Lorentz group and transform under combinations of γ-matrices. These matrices are defined on the Minkowski tangent space of a curved spacetime manifold. When working with a flat spacetime this is a trivial point since the tangent space is equivalent to the spacetime itself[5]. However, in our AdS_5 bulk space this is not the case and we must introduce machinery to connect the bulk space to the tangent space where the γ-matrices live.

3.1 Stairway to the Tangent Space

Our primary tool shall be functions called vielbeins that convert between the curved-space 'coordinate frame' indices and the Minkowski 'tangent frame' indices. We will only highlight the relevant points. A full treatment can be found in chapter 12 of Bertlmann [25] or chapter 13 of Wald [19].

In curved spacetime, the **equivalence principle** states that at any point x_0 it is always possible to choose locally inertial coordinates $X^A_{x_0}$ such that the metric is Minkowski at that point: $g_{MN}(x_0) = \eta_{MN}$. At this point alone does our coordinate system have the desirable quality of being flat and isomorphic to the Minkowski tangent frame. Hence at this point we may swap tangent frame indices with coordinate indices. Our plan is to exploit this property by 'pulling back' the metric to any other point x,

$$\begin{aligned} g_{MN} &= \eta_{AB}\; \partial_M X^A_{x_0}(x)\; \partial_M X^B_{x_0}(x) && (3.1)\\ &= \eta_{AB}\; e^A_M(x)\; e^B_N(x), && (3.2)\end{aligned}$$

where we have defined the **vielbein** $e^A_M(x) = \partial_M X^A_{x_0}(x)$, which is a kind of 'square root' of the metric. The vielbein allows us to use the flatness of the special point x_0 (or alternatively the freedom to choose any such point) to convert coordinate frame indices M, N to tangent frame indices A, B. The cost of this trick is that the vielbein is position-dependent through its dependence on the coordinate system $X^A_{x_0}$.

[5]Note also that we did not need to worry about this when we placed Standard Model on a brane since all Standard Model fields are confined to propagate in a space with a flat geometry.

Spacetime indices are raised and lowered with the bulk metric g_{MN} while the tangent space indices are raised and lowered with the Minkowski metric η_{AB}. Using this we can construct the inverse vielbein,

$$e_A^M(x) = \eta_{AB}\, g^{MN}(x) e_N^B(x). \tag{3.3}$$

One can then go on to reconstruct general relativity 'on the tangent frame,' which the keen reader may pursue the details in chapter 12 of Bertlmann [25] or the summary by Sundrum in [11]. The relevant result which we shall cite is the form of the curved space covariant derivative in the tangent frame,

$$D_M = \partial_M + \frac{1}{2}\omega_{ABM}(x)\sigma^{AB} \tag{3.4}$$

where the γ-matrices in 5D will be defined in the following section and $\sigma^{AB} = \frac{1}{4}[\gamma^A, \gamma^B]$. ω is referred to as the **spin connection** and replaces the usual Christoffel connection in the spacetime covariant derivative. The explicit form of the spin connection is quite nasty [11],

$$\omega^{AB}{}_M = \frac{1}{2} g^{KL} e_K^{[A} \partial_{[M} e_{L]}^{B]} + \frac{1}{4} g^{KL} g^{NP} e_K^{[A} e_N^{B]} \partial_{[P} e_{L]}^C e_M^D \eta_{CD}, \tag{3.5}$$

but the point for us will be that it will vanish in the RS1 covariant derivative.

3.2 5D representations of the Clifford algebra and the chirality problem

Now that we've established a bridge between spacetime and the tangent space, let us examine the γ-matrices that live on this 5D tangent space. The γ-matrices satisfy the Clifford algebra and, in five dimensions, are given by

$$\gamma^A = \begin{cases} \gamma^\mu & \text{if } A = \mu = 0, \cdots, 3 \\ -i\gamma^5 & \text{if } A = 5 \end{cases}, \tag{3.6}$$

where γ_5 is the usual fifth gamma matrix[6], $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Because these are exactly the usual 4×4 matrices used in 4D quantum field theory on flat spacetime, our 5D fermions will also be four-component spinors. γ^5, however, is no longer a 'special' element of the Clifford algebra that can be used to define a 4D parity operator. Hence there is no analogous parity operator in 5D; 5D spinors are Dirac and decompose into the $(0, 1/2) \oplus (1/2, 0)$ representation in 4D.

Naively, this means that one cannot write down a 5D theory that reduces to a chiral 4D theory. If every 4D fermion originates from a 5D Dirac spinor, then every chiral 4D fermion must be paired with a sister fermion of the opposite chirality and identical quantum numbers. This **chirality problem** is a general feature of extra dimensional models with bulk fermions [16]. It is of particular relevance since our intent is to include only a bulk right-handed neutrino at low-energies.

In Section 4.2 we will see that we can use the RS1 orbifold to get rid of these extra degrees of freedom. In fact, orbifolding is a common procedure that extra dimensional model builders can invoke to write down chiral theories. Fortunately, in RS1 we are given an orbifold 'for free' as a feature of the model. The general strategy is to eliminate parity states that cannot simultaneously satisfy the S^1 periodic boundary conditions $\mathcal{L}(x, \phi) = \mathcal{L}(x, \phi + 2\pi)$ and the $\mathbb{Z}_2$ symmetry $\mathcal{L}(x, \phi) = \mathcal{L}(x, -\phi)$. As an

[6] It is, in fact, a general feature of the representation of the Clifford algebra in $2n$-dimensions that one can define an additional 'parity' element $\gamma_{2n+1} = i\gamma_1 \cdots \gamma_{2n}$ that is also the element needed to extend to the $(2n + 1)$-dimensional representation. See, for example reference [26].

illustrative example, let us demonstrate this for the case of a flat extra dimension. Our 5D flat-space Dirac action takes the form

$$S_{\rm f} = \int d^4x \int d\phi\ \bar{\Psi} i\partial\!\!\!/ \Psi, \tag{3.7}$$

where Ψ decomposes into 4D chiral spinors $\Psi_{L,R}$ and $i\partial\!\!\!/ \equiv i\gamma^M\partial_M$ as one might naturally extend from 4D. The ϕ component of this sum contains the partial derivative $\partial_5 = \frac{\partial}{\partial\phi}$, which is odd under ϕ-parity. Hence the fields in the term $\Psi i\gamma^5\partial_5\Psi = i\Psi_L\partial_5\Psi_R - i\Psi_R\partial_5\Psi_L$ must together contribute an overall minus sign under parity to offset the sign flip of the derivative and preserve the $\mathbb{Z}_2$ symmetry. Thus we see that one of the chiral fields must be ϕ-odd while the other must be ϕ-even. In a standard KK decomposition for a flat extra dimension, however, the odd modes are proportional to $\sin(n\phi)$. Thus the zero mode of the odd chiral field vanishes and the ground state of the theory is chiral. This is a toy version of the warped case in Section 4, but a similar elimination of a zero mode chiral state will occur, though our eigenfunctions will turn out to be more complicated.

3.3 Fermionic action in a warped extra dimension

At the end of the last section we wrote the fermionic action for a flat extra dimension. In order to write the fermionic action in curved space we promote the partial derivative to a covariant derivative[7], D. Next, the tangent space index on the γ-matrices must be converted into a spacetime index using the inverse vielbein. Finally, we insert the usual factor of $\sqrt{g}$. Our fermionic action then takes the form

$$S_{\rm f} = \int d^4x \int d\phi\ \sqrt{g}\ \bar{\Psi}\ i\gamma^A e_A^M(x,\phi) D_M\ \Psi, \tag{3.8}$$

where D_M is the covariant derivative defined in equation (3.4). We would like to massage this into a more useful—if also unsightly—form. Separating out the spin connection term, we have

$$\bar{\Psi} i\gamma^A e_A^M(x,\phi) D_M\Psi = \bar{\Psi} i\gamma^A e_A^M(x,\phi) \left(\partial_M + \frac{1}{2}\omega_{BCM}\sigma^{BC}\right)\Psi. \tag{3.9}$$

We now use the following identities,

$$2\gamma^A\sigma^{BC} = [\gamma^A, \sigma^{BC}] + \{\gamma^A, \sigma^{BC}\}, \tag{3.10}$$
$$\left[\gamma^A, \sigma^{BC}\right] = \gamma^C\eta^{BA} - \gamma^B\eta^{CA}. \tag{3.11}$$

The first identity is just the statement that any product can be written as the sum of a commutator and anticommutator, while the second comes from the definition $\sigma^{BC} = \frac{1}{4}[\gamma^B, \gamma^C]$. We insert these into equation (3.9) and note that the commutator term vanishes by the antisymmetry of ω_{BCM} and the Clifford algebra. Finally, integrate by parts in the to get the following form of the fermion action [25],

$$S_{\rm f} = \int d^4x \int d\phi\ \sqrt{g}\ e_A^M(x,\phi) \left[\bar{\Psi} i\gamma^A \overleftrightarrow{\partial}_M\Psi + \frac{\omega_{BCM}}{2}\bar{\Psi} i\{\gamma^A, \sigma^{BC}\}\Psi\right] \tag{3.12}$$

where $\overleftrightarrow{\partial}_A = \frac{1}{2}(\partial_A - \overleftarrow{\partial}_A)$. This form of the action may not seem like much of an improvement, but we shall see that in RS1 the connection term vanishes.

[7]For our present purposes this is a purely spacetime-geometric covariant derivative and has nothing to do with the gauge covariant derivative used to couple fermions to gauge bosons.

4. Neutrino Masses in RS1

Now that we've established the necessary formalism to deal with fermions on curved spacetime we would like to extend our minimal RS1 model to incorporate low-energy bulk right-handed neutrinos. After some brief words of motivation, we will discuss the general formalism of a bulk fermion in RS1 and then specialize to the case of bulk neutrino. Finally, we will present a realistic model of naturally small 4D neutrino masses coming from bulk right-handed neutrinos.

4.1 Leaving braneworld

Let us begin by first motivating our choice to 'liberate' a fermionic field from the brane. Our goal is to recycle the RS1 machinery to explain the neutrino mass scale in the same way that we used it to explain the weak scale in Section 2. In order to access the exponential warping along the extra dimension we will have to work with fields propagating in the orbifold dimension.

One might be suspicious that it is ad hoc to allow some fields to propagate in the bulk while its siblings are confined to a brane. We require a right-handed neutrino to introduce a Dirac neutrino mass term in our theory. This particle, however, is special because it is a singlet with respect to the Standard Model gauge group. Hence we may suppose that there is a more fundamental theory that confines nontrivial gauge group representations to the brane. Just as we didn't concern ourselves with the exact mechanism by which our branes originated, we similarly can treat our extension of RS1 as an effective theory to such a fundamental theory.

The RS1 set up rewards our willingness to insert a bulk neutrino by providing us with a KK decomposition that allows modes that are localized on the Planck brane and hence can take advantage of the warp factor to generate a small 4D mass term.

4.2 Bulk fermions in RS1

Let's now begin to put all of these ingredients together to incorporate bulk fermions into the RS1 model. This section follows the argument set forth by Grossman and Neubert's pioneering paper on bulk neutrinos in RS1 [8], though the techniques presented are valid for general species of fermions placed in the bulk. To clarify the procedure, we shall explicitly label each step along the way.

Step 1. Insert the RS1 metric. We begin by plugging in our RS1 metric (2.16) into the machinery of the previous section. The inverse vielbein is

$$e^A_M = \mathrm{diag}(e^{-8\sigma}, e^{-8\sigma}, e^{-8\sigma}, e^{-8\sigma}, 1/r), \tag{4.1}$$

and the metric determinant is $g = r^2 e^{-8\sigma}$, where $\sigma = kr|\phi|$. Since the metric is diagonal, the only non-vanishing entries of ω_{BCM} have $B = C = M$. Thus, by the antisymmetry of ω in its first two indices, the spin connection term in (3.12) vanishes, as promised.

Step 2. Insert a mass term. When we wrote our fermionic action in curved space in Section 3.3 we did not attempt to incorporate a mass term. In the RS1 model, the orbifold symmetry prevents us from writing down the naive choice,

$$\Delta\mathcal{L}_\mathrm{f} = -m\bar{\Psi}\Psi = -m\bar{\Psi}_L\Psi_R - m\bar{\Psi}_R\Psi_L, \tag{4.2}$$

where the subscripts refer to 4D chiralities $\Psi_{\mathrm{L,R}} \equiv \frac{1}{2}(1 \mp \gamma_5)\Psi$. Recall from our discussion of the chirality problem in Section 3.2 that the $i\bar{\Psi}\gamma^5\partial_\phi\Psi$ contribution to the kinetic term forced our two chiral fields to have opposite parities, and thus (4.2) does not obey the $\mathcal{L}(\phi) = \mathcal{L}(-\phi)$ symmetry. In order to preserve this orbifold symmetry, we must insert a factor to compensate the sign flip under a ϕ-parity transformation. Our mass term must then be of the form[8]

$$\Delta\mathcal{L}_\mathrm{f} \,=\, -m \cdot \mathrm{sgn}(\phi)\bar{\Psi}\Psi. \tag{4.3}$$

The appearance of $\mathrm{sgn}(\phi)$ may seem unnatural, but we invoke the idea that our model is an effective theory. The above mass term can be generated, for example, by the vacuum expectation value of a bulk Higgs-like field that is odd under ϕ parity. The details, as usual, are irrelevant to low-energy physics.

Step 3. Write the bulk fermionic action. We now insert this mass term and the RS1 values for e_M^A and g into our fermionic action (3.12) and integrate by parts, remembering that left-acting derivatives also act on the vielbein and metric determinant. We end up with the hefty equation,

$$\begin{aligned} S_\mathrm{f} \,=\, \int d^4x \int d\phi \; r\Big\{ & e^{-3\sigma}\left(\bar{\Psi}_\mathrm{L} i\partial\!\!\!/\Psi_\mathrm{L} + \bar{\Psi}_\mathrm{R} i\partial\!\!\!/\Psi_\mathrm{R}\right) - e^{-4\sigma} m \cdot \mathrm{sgn}(\phi)\left(\bar{\Psi}_\mathrm{L}\Psi_\mathrm{R} + \bar{\Psi}_\mathrm{R}\Psi_\mathrm{L}\right) \\ & -\frac{1}{2r}\left[\bar{\Psi}_\mathrm{L}\left(e^{-4\sigma}\partial_\phi + \partial_\phi e^{-4\sigma}\right)\Psi_\mathrm{R} - \bar{\Psi}_\mathrm{R}\left(e^{-4\sigma}\partial_\phi + \partial_\phi e^{-4\sigma}\right)\Psi_\mathrm{L}\right]\Big\}. \end{aligned} \tag{4.4}$$

Step 4. Ansatz for the KK decomposition. The next step is to insert a KK decomposition in terms of 'nice' eigenfunctions. However, due to the curvature from our warp factor, it is not clear what the 'nice' eigenfunctions along our orbifold direction might be. Let us make the ansatz for the form of the KK decomposition that was proposed by Grossman and Neubert [8],

$$\Psi_{L,R}(x,\phi) \,=\, \sum_n \psi_n^{L,R}(x)\frac{e^{2\sigma}}{\sqrt{r}}\hat{f}_n^{L,R}(\phi). \tag{4.5}$$

This peculiar choice will be validated below when we change variables. We shall justify the orthogonality of the $\hat{f}$s shortly. The important feature of our decomposition is that it must reduce the 5D action (4.4) to a 4D action with a Kaluza-Klein tower of fermions,

$$S_\mathrm{f}^{(4\mathrm{D})} \,=\, \sum_n \int d^4x \left\{\bar{\psi}_n(x) i\partial\!\!\!/\psi_n(x) - m_n\bar{\psi}_n(x)\psi_n(x)\right\}, \tag{4.6}$$

where $\psi_n \equiv \psi_n^L + \psi_n^R$ and the masses $m_n \geq 0$ are naturally on the order of the weak scale by the RS1 warping mechanism. By imposing that our eigenfunction ansatz (4.5) reduces to (4.6) upon integrating out ϕ, we get the following conditions on our eigenmodes $\hat{f}_N^{L,R}(\phi)$:

$$\int d\phi \; e^{\sigma}(\hat{f}_n^{L,R})^*\hat{f}_m^{L,R} \,=\, \delta_{mn} \tag{4.7}$$

$$\left(\pm\frac{1}{r}\partial_\phi - m\right)\hat{f}_n^{L,R}(\phi) \,=\, -m_n e^{\sigma}\hat{f}_n^{R,L}(\phi). \tag{4.8}$$

Explicit calculation of these conditions is performed in Appendix B. The first of these conditions is the orthonormality condition for our eigenfunctions and justifies our ansatz in equation (4.5). The second condition is a differential equation whose solution will allow us to write down the functional form of the $\hat{f}^{L,R}$.

[8] Special thanks to Matthew Reece for an illuminating conversation regarding connecting this mass term to the chirality problem.

Old variable		'Nice' variable			Comment
5D coordinate	ϕ	t	$=$	$e^{-kr(\pi-\|\phi\|)}$	$t \in [\epsilon, 1]$; $\epsilon = e^{-kr\pi}$ is the warp factor
Bulk mass	m	ξ	$=$	$\frac{m}{k}$	Ratio of bulk mass to curvature
4D KK mass	m_n	x_n	$=$	$\frac{m_n}{\epsilon k}$	Scaled ratio of KK mass to curvature
Eigenfunction	$\hat{f}_n^{L,R}(\phi)$	$f_n^{L,R}(t)$	$=$	$\sqrt{kr\epsilon}\hat{f}_n^{L,R}(t)$	Satisfies 'nice' properties (4.11 - 4.14)

Table 1: A summary of the change to 'nice' coordinates and eigenfunctions in step 5.

In addition to these conditions, we also have boundary conditions on our fields coming from our orbifold boundary conditions. Recall from our discussion of the chirality problem in Section 3.2 that left-handed and right-handed fields must have opposite ϕ-parities. Thus, the product of odd and even ϕ-eigenfunctions must vanish at the orbifold fixed points at $\phi = 0$ and $\phi = \pm\pi$. Hence our orbifold symmetry imposes the boundary conditions

$$\hat{f}_m^L(0)^* f_n^R(0) = f_m^L(\pm\pi)^* f_n^R(\pm\pi) = 0. \tag{4.9}$$

Step 5. Change to nice variables. Continuing to follow Grossman and Neubert, let's introduce new dimensionless variables to simplify the calculations. Instead of ϕ, we will work with the variable $t \equiv e^{-kr(\pi-|\phi|)} \in [\epsilon, 1]$ where $\epsilon = e^{-kr\pi}$ is the warp factor on the visible brane and is of order 10^{-16}. Further, instead of m and m_n we will work with parameters

$$\xi = \frac{m}{k} \qquad\qquad x_n = \frac{m_n}{\epsilon k}. \tag{4.10}$$

Finally, let us scale our eigenfunctions by $\hat{f}_n^{L,R}(t) \to \sqrt{kr\epsilon} f_n^{L,R}(t)$. This change of variables is summarized in Table 1. We can now recast our conditions(4.7 - 4.9) into a more natural form,

$$\int_\epsilon^1 dt\ f_m^{L,R}(t)^* f_n^{L,R}(t) = \delta_{mn} \tag{4.11}$$

$$(\pm t\partial_t - \xi) f_n^{L,R}(t) = -x_n t f_n^{R,L}(t) \tag{4.12}$$

$$f_m^L(\epsilon)^* f_n^R(\epsilon) = 0 \tag{4.13}$$

$$f_m^L(1)^* f_n^R(1) = 0. \tag{4.14}$$

Step 6. Determine the zero mode. The zero mode is especially important for our low energy theory. Fortunately, they are also easy to solve since $m_0 = 0$ implies $x_0 = 0$ and so that the differential equations (4.12) decouple. The normalized solution is given by

$$f_0^{L,R}(t) = f_0^{L,R}(1)\ t^{\pm\xi} \propto e^{\pm mr|\phi|} \tag{4.15}$$

$$|f_0^{L,R}(1)|^2 = \frac{1 \pm 2\xi}{1 - \epsilon^{1\pm 2\xi}}. \tag{4.16}$$

Recall that in Section 3.2 we avoided the chirality problem in flat space because the boundary conditions at the orbifold fixed points forced odd zero modes (corresponding to unwanted chiralities) to vanish. In our present case a similar mechanism in place. The orbifold boundary condition (4.14) again forces one of the chiral modes to vanish. Stated in another way, the fact that both the left- and right-handed solutions are even in ϕ contradicts with the requirement that they have opposite ϕ-parities which we derived from the kinetic term.

We see that at $t = 1$, corresponding to $\phi = \pm\pi$, the right-handed eigenfunction $f_0^R(1) \propto \epsilon^{\xi-\frac{1}{2}}$. This means that for $\xi > 1/2$, the right-handed zero mode wave function is suppressed by the warp

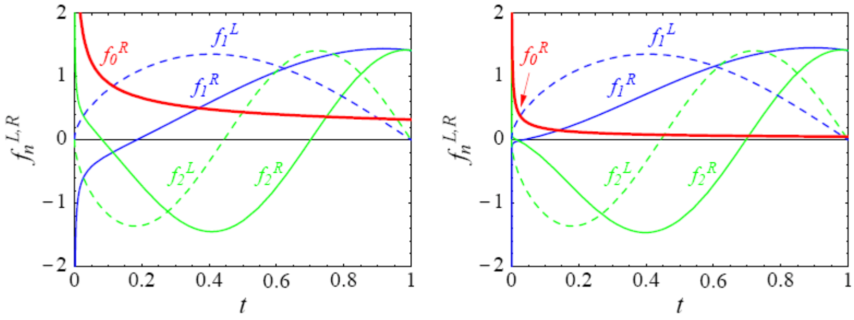

Figure 3: Examples of left- (dashed) and right-handed (solid) eigenmodes $f_n^{L,R}$ with $\xi = .45$ (left) and $\xi = .55$. Boundary conditions are set such that left-handed fields vanish at $t = \epsilon, 1$. Note that for $\xi > 1/2$ the right-handed zero mode f_0^R is strongly localized on the hidden brane $t = \epsilon$. Figure from [8].

factor at the visible brane. This is illustrated in Figure 3, which displays various left- and right-handed eigenmodes for $\xi < 1/2$ and $\xi > 1/2$. This is exactly how we will utilize the metric to 'redshift' our neutrino masses in the next section.

Step 7. Determine KK modes. With the zero modes understood we move on to determine the higher KK excitations. We can combine the equations in (4.12) into a single second-order equation,

$$\left[t^2\partial_t^2 + x_n^2 t^2 - \xi(\xi \mp 1)\right] f_n^{L,R}(t) = 0. \tag{4.17}$$

The general solutions are Bessel functions, which we can write as[9]

$$f_n^{L,R}(t) = \sqrt{t}\left[a_n^{L,R} J_{\frac{1}{2}\mp\xi}(x_n t) + b_n^{L,R} J_{-\frac{1}{2}\pm\xi}(x_n t)\right]. \tag{4.18}$$

Imposing our original first-order equations (4.12) as constraints, we see that $b_n^L = a_n^R$ and $b_n^R = -a_n^L$. So our final form for the eigenfunctions are

$$f_n^L(t) = \sqrt{t}\left[a_n^L J_{\frac{1}{2}-\xi}(x_n t) + a_n^R J_{-\frac{1}{2}+\xi}(x_n t)\right] \tag{4.19}$$

$$f_n^R(t) = \sqrt{t}\left[a_n^R J_{\frac{1}{2}+\xi}(x_n t) + a_n^L J_{-\frac{1}{2}-\xi}(x_n t)\right]. \tag{4.20}$$

Next we impose our orbifold boundary conditions (4.13 - 4.14), coming from our requirement that the two chiralities have opposite ϕ-parity. This gives us a discrete KK spectrum. We are free to choose either the left-handed modes or the right-handed modes to vanish at the orbifold fixed points $t = 0, \epsilon$. Since we noted in step 6 that the right-handed zero mode with $\xi > 1/2$ gave us the warping that we wanted, we shall set $f_n^L(\epsilon) = f_n^L(1) = 0$.

Our eigenfunctions will be integrated over distributions that are nonsingular in the small parameter ϵ. Thus we can determine the coefficients $a_n^{L,R}$ removing terms that diverge in the limit $\epsilon \to 0$. We find that the properly normalized coefficients are $a_n^L = 0$ and $|a_n^R|^2 = 2/|J_{1/2+\xi}(x_n)|^2$.

4.3 Generalized see-saw from a bulk neutrino

In the previous section we have developed the complete formalism for a bulk RS1 fermion localized on the hidden brane. Let us now couple this fermion to the Standard Model to develop a model for

[9]We are actually making the assumption $\xi \neq \frac{1}{2} + N$ with $N \in \mathbb{Z}$. The special cases where this does not hold lead to a decomposition in Bessel functions of the first and second kind, but this case is phenomenologically uninteresting.

neutrino mass. This section continues the procedure used by Grossman and Neubert [8]. Our bulk neutrino has lepton number $L = 1$, so that the only gauge-invariant coupling on the brane is given by a Yukawa term

$$S_{\rm H} = -\int d^4x\ \sqrt{g_{\rm vis}}\ \left\{\hat{y}\bar{\ell}_0(x)\bar{H}_0(x)\Psi_R(x,\pi) + \text{h.c.}\right\}. \tag{4.21}$$

Here the subscript 0 labels fields that have not yet been canonically normalized with respect to the brane action. Recall that the lepton doublet ℓ_0 is left-handed and has its spinor indices contracted with the bulk neutrino. Thus Ψ_L does not couple to the Standard Model since $\bar{\ell}_L\Psi_L = 0$. This is especially nice because in a high energy theory the RS1 branes would have a finite width leading to overlap with the KK tower of left-handed bulk neutrinos, and hence weak-scale neutrino masses. Let us now canonically normalize our fields as in Section 2.5,

$$S_{\rm H} = -\sum_{n\geq 0}\int d^4x\ \left\{y_n\bar{\ell}(x)\bar{H}(x)\psi_n^R(x,) + \text{h.c.}\right\}, \tag{4.22}$$

$$y_n = \sqrt{k}\hat{y}\ f_n^R(1) \equiv z f_n^R(1). \tag{4.23}$$

The second equation defines the effective Yukawa coupling y_n to absorb the rescaling of the right-handed neutrino. The combination $z \equiv \sqrt{k}\hat{y}$ is dimensionless and naturally $\mathcal{O}(1)$. After electroweak symmetry breaking, the uncharged component of the Higgs doublet gets a vacuum expectation value v and the Yukawa term (4.21) becomes a neutrino mass term $\bar{\psi}_L^\nu \mathbb{M}\ \psi_R^\nu + \text{h.c.}$ in the basis $\psi_L^\nu = (\nu_L, \psi_1^L, \cdots, \psi_n^L)$ and $\psi_\nu^R = (\psi_0^R, \cdots, \psi_n^R)$, with

$$\mathbb{M} = \begin{pmatrix} vy_0 & vy_1 & \cdots & vy_n \\ 0 & m_1 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & m_n \end{pmatrix}. \tag{4.24}$$

Here $n \to \infty$ and we have included the Dirac mass terms of the bulk neutrino. Recall that for the phenomenologically interesting case $\nu > 1/2$, $|vy_0| \sim |f_0^R(1)| \ll 1$ and we get a light neutrino from parameters that take on natural values.

Physical neutrinos have (mass)^2 given by the eigenstates of $\mathbb{M}\mathbb{M}^\dagger$. Define a unitary matrix U such that $U^\dagger\mathbb{M}\mathbb{M}^\dagger U$ is diagonal and such that the mass eigenstates $\psi_L^{(\rm phys)}$ are given by $\psi_L^\nu = U^{-1}\psi_L^{(\rm phys)}$. The neutrino mixing angle θ_ν is then defined such that $\nu_L = \cos\theta_\nu\nu_L^{(\rm phys)} + \cdots$, where $\nu_L^{(\rm phys)}$ is the lightest physical neutrino. The mass of the lightest neutrino m_ν is

$$m_\nu = v|y_0|\cos\theta_\nu \tag{4.25}$$

$$= v|zf_0^R(1)|\cos\theta_\nu, \tag{4.26}$$

where we have used (4.23) to write y_0 in terms of $f_0^R(1)$ and $z \sim \mathcal{O}(1)$. We shall see that experimental constraints force the mixing angle to be small, so that $\cos\theta_\nu \sim 1$. We wrote $|f_0^R(1)|$ explicitly in equation (4.16), from which we have

$$|f_0^R(1)| = \sqrt{(2\xi-1)(1-\epsilon^{1-2\xi})^{-1}} \tag{4.27}$$

$$\approx \sqrt{2\xi-1}\epsilon^{\xi-\frac{1}{2}}. \tag{4.28}$$

Plugging this into equation (4.26), we estimate the lightest neutrino mass to be on the order of

$$m_\nu \sim M\left(\frac{v}{M}\right)^{\xi+\frac{1}{2}}. \tag{4.29}$$

This is a powerful result that defines the neutrino mass in terms of the Planck-weak hierarchy. We have, in a sense, generalized the usual see-saw mechanism which is reproduced for the special case $\xi = \frac{3}{2}$. We can generate the phenomenologically relevant range of m_ν between 10^{-5} eV and 10 eV with ξ only taking values between 1.1 and 1.5. Once again we witness the miracle of the warp factor: we are able to construct the weak-neutrino mass hierarchy with bulk dimensionful parameters $\mathcal{O}(M)$ and with dimensionless parameters $\mathcal{O}(1)$, i.e. with all parameters taking only natural values.

Now armed with a mechanism to redshift our neutrino masses as we had hoped, let us now flesh out a more realistic model that accounts for the three generations of observed neutrinos and the constraints on the mixing angle.

4.4 Two bulk neutrinos are better than one

The first hints of neutrino mass came from solar neutrino oscillations. It is only fitting, then, that any realistic theory of neutrino masses should properly take into account the mixing between the three low energy neutrinos. The intuitive extension to three bulk neutrinos fails since the ϕ-parity of our action is broken by quantum effects when there are an odd number of bulk fermions [27, 28]. In order for a model to be anomaly-free, then, it must include an even number of bulk fermions. It turns out that only two bulk fermions are required to accommodate all experimental parameters of neutrino mass and mixing. This section will again follow the analysis of Grossman and Neubert [8].

Write the two bulk neutrinos as Ψ_1^R and Ψ_2^R. The bulk masses $m_1 > m_2$ are both on the order of the fundamental Planck scale. By analogy to equation (4.23), define the effective Yukawa couplings of the two right-handed zero modes by $y^{i\alpha} = z^{i\alpha}\epsilon^{\xi_\alpha - \frac{1}{2}}$, where $\alpha = 1, 2$ labels the two types of bulk neutrino, $i = e, \mu, \tau$ is a flavor index, and ξ_α is defined by m_α/k. As before $z^{i\alpha}$ is dimensionless and $\mathcal{O}(1)$. Note that we already have a hint the origin of the neutrino mass hierarchy since the Yukawa couplings of the two zero modes differ by a factor of order $\mathcal{O}(\epsilon^{\xi_1-\xi_2})$.

Consider the neutrino mass sub-matrix in the brane Lagrangian coming from the left-handed brane neutrinos and the two right handed zero modes: $\bar{\psi}_L^\nu \mathbb{M}_0 \psi_R^\nu$ + h.c. with $\psi_L^\nu = (\nu_e^L, \nu_\mu^L, \nu_\tau^L)$, $\psi_R^\nu = (\psi_0^{R,1}, \psi_0^{R,2})$, and

$$\mathbb{M}_0 = \begin{pmatrix} v\epsilon^{\nu_1-\frac{1}{2}}z^{e1} & v\epsilon^{\nu_2-\frac{1}{2}}z^{e2} \\ v\epsilon^{\nu_1-\frac{1}{2}}z^{\mu 1} & v\epsilon^{\nu_2-\frac{1}{2}}z^{\mu 2} \\ v\epsilon^{\nu_1-\frac{1}{2}}z^{\tau 2} & v\epsilon^{\nu_2-\frac{1}{2}}z^{\tau 2} \end{pmatrix} \tag{4.30}$$

We then diagonalize the 3×3 matrix $\mathbb{M}_0\mathbb{M}_0^\dagger$ to leading order in ϵ to find that the spectrum of physical neutrino masses is given by a massless left-handed neutrino ν_1, and two light neutrinos with masses given by

$$m_{v_1}^2 = v^2\epsilon^{2\nu_1-1}\frac{|[e\mu]|^2+|[\mu\tau]|^2+|[\tau e]|^2}{|z^{e2}|^2+|z^{\mu 2}|^2+|z^{\tau 2}|^2} \sim M^2\left(\frac{v}{M}\right)^{2\nu_1+1} \tag{4.31}$$

$$m_{v_2}^2 = v^2\epsilon^{2\nu_2-1}\left(|z^{e2}|^2+|z^{\mu 2}|^2+|z^{\tau 2}|^2\right) \sim M^2\left(\frac{v}{M}\right)^{2\nu_2+1}, \tag{4.32}$$

where we have used the notation $[ij] \equiv z^{i1}z^{j2} - z^{j1}z^{i2}$.

Now consider the **Maki-Nakagawa-Sakata** (MNS) mixing matrix between neutrino flavor and mass states. We can write the low-energy flavor eigenstates in terms of propagating states, $\nu_f = \sum_i U_{fi}\nu_i$, where i runs over the three mass eigenstates. From the diagonalization of the mass matrix

Measured Parameter	Ref.	Constraint	Latest Experimental Probe
$\tan^2\theta_\nu = 0.005 \pm 0.003$	[34]	$v_0 z/k \leq 0.1$	Invisible width of the Z^0 at LEP
$\Delta m^2_{21} \sim 8 \cdot 10^{-5}$ eV^2	[35]	$\xi_1 \approx 1.34 - 1.37$	SNO (solar), KamLAND (reactor)
$\Delta m^2_{32} \sim 2.4 \cdot 10^{-3}$ eV^2	[36]	$\xi_2 \approx 1.27 - 1.29$	Super Kamiokande (atmospheric)
$\lvert U_{e3}\rvert^2 \lesssim$ few %	[37,38]	$\lvert z^e_2\rvert < \lvert z^\mu_2\rvert, \lvert z^\tau_2\rvert$	CHOOZ and Super Kamiokande
$\sin^2 2\theta_{23} > 0.92$	[36]	See equation (4.37)	Super Kamiokande (atmospheric)
$\sin^2 2\theta_{12} = 0.86$	[35]	See equation (4.38)	SNO (solar), KamLAND (reactor)
$\sin^2 2\theta_{13} < 0.05$	[39]	See Table 3	CHOOZ(reactor)

Table 2: Experimental constraints of the Grossman-Neubert model. The results here are updated with the latest data, summaries are available in [40] and [33].

above and taking the reasonable limit $\epsilon \to 0$, we find

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} \frac{[\mu\tau]^*}{N_1} & \frac{z_2^{\mu *}[e\mu] - z_2^{\tau *}[\tau e]^*}{N_1 N_2} & \frac{z_2^e}{N_2} \\ \frac{[\tau e]^*}{N_1} & \frac{z_2^{\tau *}[\mu\tau] - z_2^{e *}[e\mu]^*}{N_1 N_2} & \frac{z_2^\mu}{N_2} \\ \frac{[e\mu]^*}{N_1} & \frac{z_2^{e *}[\tau e] - z_2^{\mu *}[\mu\tau]^*}{N_1 N_2} & \frac{z_2^\tau}{N_2} \end{pmatrix} \tag{4.33}$$

where $N_1^2 = |[e\mu]|^2 + |[\mu\tau]|^2 + |[\tau e]|^2$ and $N_2 = |z_2^e|^2 + |z_2^\mu|^2 + |z_2^\tau|^2$. Note that each of the z^i_αs are $\mathcal{O}(1)$, so the elements of U are each order unity. Hence the neutrino mixing matrix lacks the strong hierarchy of the analogous CKM matrix for quarks.

4.5 Realistic phenomenology

Let's move on to experimental constraints. A summary of relevant experimental constraints is presented in Table 2. Results in this table have been updated to include new data since the original analysis in the 1999 Grossman and Neubert paper [8].

There are three types of neutrino mixing data: the flux of solar electron neutrinos compared to the expected value from hydrogen fusion in the sun, the ratio of muon to electron neutrino flux from atmospheric neutrinos, and the oscillation of electron antineutrinos from nuclear reactors. Observations are based on detection of characteristic Cerenkov radiation in heavy water. The three sources of neutrino data are complimentary and constrain different mixing angles and mass differences. Since the publication of Grossman and Neubert's original paper, the Sudbury Neutrino Observatory and KamLAND reactor experiment have confirmed the large mixing angle MSW model of solar neutrino oscillations [29–33], so we shall disregard other scenarios that Grossman and Neubert also considered. At the end of this essay we briefly mention the current status of low energy sterile neutrinos from the LSND and MiniBooNE collaborations.

First let us consider the neutrino mixing angle between light and heavy modes for a single bulk neutrino in equation (4.26), which is given in terms of the parameters in the mass matrix (4.24) via

$$\tan^2\theta_\nu = \sum_{n\geq 1} \frac{v^2|y_n|^2}{m_n^2} = \frac{v^2|z|^2}{\epsilon^2 k^2} \sum_{n=1}^{\infty} \frac{2}{x_n^2} = \frac{1}{2\xi+1}\frac{v_0^2 z^2}{k^2}, \tag{4.34}$$

where x_n are the roots of $J_{\nu-\frac{1}{2}}(x_n) = 0$. This mixing angle is experimentally constrained to be small from the measurement of the invisible width of the Z^0 boson, which constrains the number of light neutrinos $n_\nu = 2.985 \pm 0.008$ [34]. If all three light neutrinos contain equal mixings of heavy neutrinos, then $n_\nu = 3\cos^2\theta_\nu$, from which we extract the result $\tan^2\theta_\nu = 0.005$ in table 2. Assuming ξ takes a natural $\mathcal{O}(1)$ value, this value for $\tan^2\theta_\nu$ requires $v_0 z/k \leq 0.1$, which does not require 'too much' fine tuning. However, natural predictions from bulk RS1 neutrino models are at the edge of experimental bounds and will be tested by future precision experiments.

Next, let us move on to the 'realistic' model with two bulk neutrinos and consider consistency with the neutrino mass hierarchy. Assuming that the solar and atmospheric neutrino anomalies are explained by neutrino flavor oscillation, experiments constrain the mass-squared differences Δm^2_{ij} between the light neutrinos[10]. Constraints from solar and atmospheric neutrino data are given in table 2. In our model with two bulk neutrinos, the lightest neutrino is massless. Thus $\Delta m^2_{21} = m^2_{\nu_2}$ and $\Delta m^2_{32} \simeq m^2_{\nu_3}$. Using equations (4.31-4.32), we are able to use this to constrain our see-saw parameters $\xi_{1,2}$. The key result is that $\xi_{1,2}$ take natural $\mathcal{O}(1)$ values with $\xi_{1,2} > 1/2$, as we required for bulk neutrinos localized on the hidden brane.

Finally, consider the low energy MNS matrix U in (4.33). Reactor and atmospheric data limit $|U_{e3}|^2$ to be on the order of a few percent [37,38], and so $|z_2^e|$ should be smaller than $|z_2^\mu|$ and $|z_2^\tau|$. In the limit where $|z_2^e|^2 \ll |z_2^\mu|^2 + |z_2^\tau|^2$, then the angles θ_{12} and θ_{23} satisfy:

$$\sin^2\theta_{12} \simeq \frac{4|z_1^e|^2(|z_2^\mu|^2 + |z_2^\tau|^2)|[\mu\tau]|^2}{\left[|z_1^e|^2(|z_2^\mu|^2 + |z_2^\tau|^2) + |[\mu\tau]|^2\right]^2} \tag{4.35}$$

$$\sin^2\theta_{23} \simeq \frac{4|z_2^e|^2|z_2^\tau|^2}{\left(|z_2^\mu|^2 + |z_2^\tau|^2\right)^2}. \tag{4.36}$$

The atmospheric anomaly suggests a large $\nu_\mu \leftrightarrow \nu_\tau$ mixing so that $\sin^2 2\theta_{23} > 0.92$ [36]. In the limit where $|z_2^e|$ is much smaller than the other two couplings, this constrains

$$0.64 < |z_2^\mu/z_2^\tau| < 1.57. \tag{4.37}$$

Similarly, solar and reactor neutrino data imply $\sin^2 2\theta_{12} = 0.86$ [35], implying

$$\frac{|z_1^e|}{|z_1^\mu z_2^\tau - z_1^\tau z_2^\mu|}\sqrt{|z_2^\mu|^2 + |z_2^\tau|^2} = 0.74. \tag{4.38}$$

These results are reasonable within the umbrella of naturalness.

5. Lepton Flavor Violation from bulk RS1 neutrinos

Thus far our RS1 model with bulk neutrinos has agreed nicely with experimental data while maintaining naturalness in its parameters. Theories with massive Dirac neutrinos however, are strictly constrained by upper bounds on lepton flavor violating processes. These processes are suppressed well below experimental sensitivity in Standard Model by the **GIM mechanism**[11], but are amplified in extensions to the Standard Model with massive neutrinos [42]. Lepton flavor violating effects for bulk neutrinos in RS1 were first calculated by Ryuichiro Kitano, who showed that the Grossman-Neubert model must be finely-tuned in the Higgs sector in order to meet experimental lepton flavor violation constraints [9]. Kitano calculated the branching ratio of $\mu \to e\gamma$ (see Figure 4) and the analogous radiative decays of the τ. These decays are given by loop diagrams that are summed over all intermediate Kaluza-Klein modes.

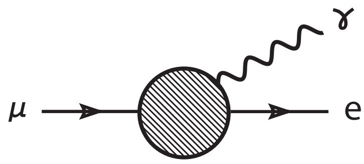

Figure 4: Heavy neutrinos violate lepton flavor conservation through radiative decay of charged leptons, such as $\mu \to e\gamma$. This process is calculated to one-loop order in Appendix C, the result is equation (5.1).

5.1 Radiative charged lepton decay

Kitano's calculation of $\mu \to e\gamma$ in the Grossman-Neubert model is done in more detail in Appendix C. The diagrams that contribute to this process at one loop are shown in Figure 5, where a sum over massive neutrinos is implied. The result for $\mu \to e\gamma$ is given by [9, 10]

$$\mathcal{M} = e\frac{g^2 m_\mu}{(4\pi)^2 M_W^2}\left(\sum_i^\infty U_{\mu i}^* U_{ei} F\left(\frac{m_i^2}{M_W^2}\right)\right)\epsilon_\lambda^*(q)\bar{u}_e(p-q)\left[i\sigma^{\lambda\rho}q_\rho(1+\gamma_5)\right]u_\mu(p) \tag{5.1}$$

$$F(z) = \frac{1}{12(1-z)^4}\left(10-43z+78z^2-49z^3-18z^3\ln z+4z^4\right), \tag{5.2}$$

where i sums over the N^{th} KK mode with $N \to \infty$. Note that Kitano's published result [9] writes this formula incorrectly[12]. F diverges when $m_i = M_W$, but we expect the light physical modes to have $m_\nu \ll M_W$ and the heavy modes to have $m_i \gg M_W$. The asymptotic values for F are given by $F(0) = 5/3$ and $F(\infty) = 2/3$.

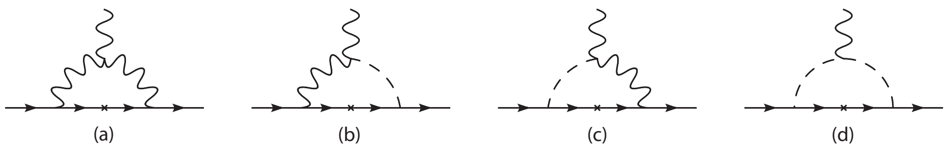

Figure 5: Diagrams that contribute to the $\mu \to e\gamma$ amplitude in equation (5.1). It is implied that one sums over all neutrino Kaluza-Klein modes in the loop. See Appendix C for more details.

The GIM mechanism suppresses diagrams mediated by light neutrinos $m_i \ll M_W$. However, this suppression fails in the opposite limit where the Kaluza-Klein excitations are very heavy, $m_i \gg M_W$. In fact, one might be especially concerned because $F(\infty) \neq 0$ and the sum over the KK tower appears to diverge. In this limit, however, we are saved by the **decoupling** of very heavy states from low-energy processes, i.e. the fact that physics at very small scales shouldn't affect physics at much larger scales [45]. Pedagogically this is because processes mediated by a heavy particle are mass suppressed by their propagators. Kitano demonstrates decoupling analytically by looking at the parenthesis in

[10] By convention we take $\Delta m_{ij}^2 = m_{\nu_i}^2 - m_{\nu_j}^2$ where $m_{\nu_1} < m_{\nu_2} < m_{\nu_3}$

[11] In a nutshell, the GIM mechanism describes why flavor changing processes cannot occur at tree level by noting that such amplitudes are proportional to $\sum_k UikU_{jk}^* = 0$, where U are unitary mixing matrices [41]. Further, such processes are suppressed at loop level by $\delta m^2/M_W^2$, where, in our case, $\delta m = m_\nu$.

[12] Kitano's equation (20) forgot the power of four in the denominator which is crucial for convergence when summing over KK modes. He also did not cite Cheng and Li's calculation in [43, 44]. I have e-mailed him regarding these points.

equation 5.1) and setting $F(m_i^2/M_W^2) \to F(\infty)$. By using equation (4.34) and $M_W \sim gv$, one can approximate the remaining summed term as

$$\frac{g^2}{M_W^2}\sum_i U_{ei}U_{\mu i}^* \approx \sum_n \left(\frac{z_{e1}z_{\mu 1}}{m_{n,1}^2} + \frac{z_{e1}z_{\mu 2}}{m_{n,2}^2}\right) \tag{5.3}$$

$$= \frac{1}{(k\epsilon)^2}\sum_n \left(z_{e1}z_{\mu 1}\frac{1}{x_{n,1}^2} + z_{e2}z_{\mu 2}\frac{1}{x_{n,2}^2}\right) \tag{5.4}$$

$$= \frac{1}{(k\epsilon)^2}\left(\frac{z_{e1}z_{\mu 1}}{4\xi_1+2} + \frac{z_{e2}z_{\mu 2}}{4\xi_2+2}\right). \tag{5.5}$$

The final line is finite, as promised, though it doesn't provide much physical intuition about the source of the decoupling. Cheng and Li followed up Kitano's calculation by explaining that $U_{eA}U_{\mu A}^*$ is proportional to $\sin^2\theta_A$, the mixing of the low energy states with the heavy state A [10]. From (4.34) and experimental limits we see that this mixing gives the desired decoupling $\sin^2\theta_A \sim (m_\nu/m_A)^2$.

5.2 Lepton flavor violation phenomenology

Making use of the approximation (5.5), decay width for radiative muon decay is given by

$$\Gamma(\mu \to e\gamma) = \frac{e^2 m_\mu^5}{4(4\pi)^5(\epsilon k)^4}\left|\frac{z_{e1}z_{\mu 1}}{4\xi_1+2} + \frac{z_{e2}z_{\mu 2}}{4\xi_2+2}\right|^2 \tag{5.6}$$

$$= 0.0037\left(\frac{v}{\epsilon k}\right)^2\left|\frac{z_{e1}z_{\mu 1}}{4\xi_1+2} + \frac{z_{e2}z_{\mu 2}}{4\xi_2+2}\right|^2. \tag{5.7}$$

The analogous τ decays are given by making the replacements $\mu \to \tau$, $e \to \mu, e$ and $0.0037 \to 0.00065$. The LAMPF-MEGA and CLEO experiments bound the branching ratios of these decays to be [46–48]

$$\mathrm{Br}(\mu \to e\gamma) < 1.2 \times 10^{-11} \tag{5.8}$$

$$\mathrm{Br}(\tau \to \mu\gamma) < 1.1 \times 10^{-6} \tag{5.9}$$

$$\mathrm{Br}(\tau \to e\gamma) < 2.7 \times 10^{-6}. \tag{5.10}$$

Thus, using the fact that z and ξ are $\mathcal{O}(1)$, one can constrain

$$\frac{v}{\epsilon k} \lesssim 0.02. \tag{5.11}$$

Applying relation (4.10), this gives us a bound on the lowest Kaluza-Klein mass,

$$m_{KK} \gtrsim 25 \text{ TeV}. \tag{5.12}$$

This is two orders of magnitude greater than the Higgs vacuum expectation value, so a fine tuning on the order of 10^{-2} is necessary. We can go a bit further with this and recast the constraints on the mixing angles in table 2 in terms of the $\mathcal{O}(1)$ Yukawa parameters using (4.34). We first employ the approximate relations between dimensionless Yukawa couplings

$$|z_1^e| \sim 0.7|z_1^\mu| \sim 0.7|z_1^\tau|. \tag{5.13}$$

Using this, Grossman and Neubert computed more stringent constraints on the Yukawa couplings, presented in table 3. These couplings are also pushing the limits of what one might consider natural.

Family index, i	**Bulk index,** α	**Constraint on** $\lvert z^i_\alpha \rvert$	**Conditions**
e, μ, τ	1	$\lesssim 0.02$	Large angle MSW solution
e	2	$\lesssim 0.009$	Taking $\sin^2 2\theta_{13} = 0.05$ [39]
μ, τ	2	$\lesssim 0.05$	Taking $\sin^2 2\theta_{13} = 0.05$ [39]

Table 3: Constraints on the effective dimensionless Yukawa couplings $|z^i_\alpha|$ from the $\mu \to e\gamma$ branching ratio and the mixing angle constraints in table 2.

6. Summary and further directions

In this essay we have introduced the RS1 model and the Grossman-Neubert minimal extension with bulk fermions to explain neutrino masses and mixings. This extension is able to use the RS1 warp factor to generate the neutrino-weak hierarchy and the hierarchy between neutrino masses via a generalized see-saw mechanism with only natural parameters. We then focused on experimental constraints from lepton flavor violation in charged lepton decay. This process was able to strictly constrain the lowest KK mass and Yukawa couplings of the bulk neutrino, but it forced the minimal model into a corner by requiring levels of fine-tuning. The next source of precision data in the lepton sector would come from the proposed International Linear Collider (ILC), which will further probe lepton flavor violating processes [49].

Until then, the model discussed in this essay lends itself to further directions of research. The formalism developed for bulk RS1 neutrinos is general and can be applied to other fermionic particles. In particular, Gherghetta and Pomarol have proposed a supersymmetric RS1 where all fields are free to propagate in the bulk [50]. During the course of this writing [51] appeared. It studies $\mu \to e\gamma$ decays with a bulk Standard Model and two Higgs doublets. One can go further and place all sorts of Standard Model extensions into the bulk, such as technicolor. Because of its strong suppression in the Standard Model and sensitivity to particles that couple to leptons and the W boson, lepton flavor violation is a strong experimental check for physics beyond the Standard Model. Like many signals, unfortunately, it is difficult to disambiguate between models of extra dimensions and supersymmetry [52] (or even more exotic models).

One can also continue from the work presented here in one of many orthogonal directions. A recent constraint on particle physics models comes from early universe cosmology. Studies of the cosmology of warped extra dimensions [53] can be used in conjunction with dark matter densities to constrain the relic density of undiscovered heavy particles. In another direction, the general flavor structure of models with a warped extra dimension have only recently been studied [54]. To the best of this author's knowledge, there is yet no published analysis of CP violation in the Grossman-Neubert model. This may be an interesting topic to study that may yield some insight on CP violation beyond the Standard Model. Yet another direction involves recent attempts in bottom-up neutrino model-building by Ma and his collaborators. Ma has proposed of a discrete $A4$ family symmetry which seems to approximate observed neutrino mixing results [55]. One might attempt to connect the theoretical prediction of RS1 bulk neutrino mixing to this $A4$ family symmetry or, conversely, use the $A4$ symmetry to motivate non-minimal bulk neutrino extensions to RS1. Again, to the best of this author's knowledge, there is yet no published analysis of the compatibility of the Grossman-Neubert model with such bottom-up predictions.

Finally, the Grossman-Neubert model with two bulk neutrinos would have to be extended to at least four neutrinos if the lightest neutrino is found to have nonzero mass. This, however, would naively suggest the existence of a light sterile neutrino. Recent results from the MiniBooNE collaboration, however, appear to refute LSND results that suggested the existence of such a particle [56]. To close, I offer a quote from the anonymous CERN blog Resonaances[13] regarding the possibility of accommodating both LSND and MiniBooNE data within an extra dimensional model: "*By shaping the extra dimensions properly, the model can accommodate the MiniBooNE excess at lower energies as well as the suppression of oscillations at higher energies. My impression is that, with a little bit more work, the model could accommodate Harry Potter, too.*"

7. Acknowledgements

I would like to thank Ben Allanach for his time guiding this essay and for many illuminating conversations on particle phenomenology. I credit Matthew Reece for connecting the 5D chirality problem with the sign the bulk fermion mass term. After the writing of this essay Matthias Neubert also offered his thoughts on the chirality problem in a personal communication. LaTeX typesetting was done using MiKTeX 2.5, Latex Editor (LEd), and Jaxodraw [57]. I would like to thank my fellow students Steffen Gielen, Leo van-Nierop, and David Simmons-Duffin for letting me bounce ideas off them while brainstorming for this essay. Additional thanks to Steffen Gielen for proof reading my essay for spelling and grammar. Finally, I would like to dedicate this work to all of my Part III friends from whom I have learned so much and whose shared passion for physics (and late night table football) has made Part III a very special experience.

A. Notation and Convention

There is no globally accepted standard for conventions when dealing with extra dimensions. I will try to minimize the redefinition of common notation. The effective 4D Planck scale is denoted by the usual M_{Pl} while the fundamental 5D Planck scale is M. Bulk coordinates are indexed with capital Roman letters from the middle of the alphabet, $M, N \in \{0, \cdots, 4\}$. We parameterize the extra dimension x^5 by an angular coordinate, ϕ, with an associated radius of compactification r. Cartesian coordinates on the 3-brane are given by the usual lowercase Greek letters, $\mu, \nu \in \{0, \cdots, 3\}$. The bulk metric is denoted by g_{MN} and the 3-brane metric denoted by $(g_{\text{vis}})_{\mu\nu}$.

When dealing with fermions it is necessary to work with a little more formalism and distinguish between the manifold and its tangent space. We shall reserve capital Roman letters (A, B) at the beginning of the alphabet to index the bulk tangent space. The bulk vielbein is given by e_M^A. The inverse vielbein is denoted with flipped indices, e_A^M. We will take the Minkowski metric to be 'mostly minus',

$$\eta_{AB} = \text{diag}(+,-,-,-,-) \tag{A.1}$$

$$\eta_{\alpha\beta} = \text{diag}(+,-,-,-) \tag{A.2}$$

In the interests of readability, I have taken the liberty of using different variables relative to some of the main works cited. As a final point of convention, I apologize for adopting American English and for its brutalization of 'proper' spelling.

[13] http://resonaances.blogspot.com/2007/04/after-miniboone.html

B. Properties of the RS1 bulk eigenbasis

Here we derive the orthogonality relation (4.7) and differential equation (4.8) for the $\hat{f}(\phi)$ eigenbasis of the RS1 orbifold direction. These follow from demanding that integrating out ϕ in the bulk action gives us the effective action. Recall that the bulk action takes the form (4.4):

$$\begin{aligned} S_{\mathrm{f}} = & \int d^4x \int d\phi\, r \Big\{ e^{-3\sigma} \left(\bar{\Psi}_{\mathrm{L}} i\partial\!\!\!/ \Psi_{\mathrm{L}} + \bar{\Psi}_{\mathrm{R}} i\partial\!\!\!/ \Psi_{\mathrm{R}} \right) - e^{-4\sigma} m \cdot \mathrm{sgn}(\phi) \left(\bar{\Psi}_{\mathrm{L}} \Psi_{\mathrm{R}} + \bar{\Psi}_{\mathrm{R}} \Psi_{\mathrm{L}} \right) \\ & - \frac{1}{2r} \left[\bar{\Psi}_{\mathrm{L}} \left(e^{-4\sigma} \partial_\phi + \partial_\phi e^{-4\sigma} \right) \Psi_{\mathrm{R}} - \bar{\Psi}_{\mathrm{R}} \left(e^{-4\sigma} \partial_\phi + \partial_\phi e^{-4\sigma} \right) \Psi_{\mathrm{L}} \right] \Big\}. \end{aligned} \tag{B.1}$$

We compare this to the effective action for the Kaluza-Klein tower (4.6):

$$S_{\mathrm{f}}^{(4\mathrm{D})} = \sum_n \int d^4x \left\{ \bar{\psi}_n(x) i\partial\!\!\!/ \psi_n(x) - m_n \bar{\psi}_n(x) \psi_n(x) \right\}, \tag{B.2}$$

Finally, recall that the our Kaluza-Klein decomposition is (4.5):

$$\Psi_{L,R}(x,\phi) = \sum_n \psi_n^{L,R}(x) \frac{e^{2\sigma}}{\sqrt{r}} \hat{f}_n^{L,R}(\phi). \tag{B.3}$$

B.1 Orthogonality

The 4D derivative terms should give us that corresponding kinetic terms in the effective theory. Looking at the left handed components of the these terms, we get

$$r \left\{ e^{-e\sigma} \left(\bar{\psi}_L \; i\partial\!\!\!/ \; \psi_L \right) \right\} = r e^{-e\sigma} \left\{ \left(\sum_n \bar{\psi}_n^L \frac{e^{2\sigma}}{\sqrt{r}} \hat{f}_n^{L*} \right) i\partial\!\!\!/ \left(\sum_m \psi_n^L \frac{e^{2\sigma}}{\sqrt{r}} \hat{f}_n^L \right) \right\} \tag{B.4}$$

$$= r e^{-3\sigma} \sum_{n,m} \left(\frac{e^{4\sigma}}{r} \hat{f}_n^{L*} \hat{f}_m^L \; \bar{\psi}_n^L \; i\partial\!\!\!/ \; \psi_n^L \right) \tag{B.5}$$

$$= e^{\sigma} \hat{f}_n^{L*} \hat{f}_m^L \tag{B.6}$$

The right handed part is the same with $L \to R$. Integrating and comparing to the kinetic term of (4.6), we see that we get the orthnormality condition (4.7),

$$\int d\phi\; e^{\sigma} (\hat{f}_n^{L,R})^* \hat{f}_m^{L,R} = \delta_{mn}. \tag{B.7}$$

B.2 Differential Equation for $\hat{f}$

Now consider the remaining terms in the bulk action (4.4). These are (anti)symmetric with respect to $(L \leftrightarrow R)$, so it is sufficient to work with terms of only one chirality. Because of the orbifold symmetry it is sufficient to assume $\phi > 0$. First consider the term coming from the ϕ-component of the 5D kinetic part of the Lagrangian:

$$-\frac{1}{2}\bar{\Psi}_L\left\{e^{-4\sigma},\partial_\phi\right\}\Psi_R = -\sum_{m,n}\frac{1}{2}(\bar{\psi}^L_n\frac{e^{2\sigma}}{\sqrt{r}}\hat{f}^{L*}_n)(e^{-4\sigma}\partial_\phi+\partial_\phi e^-4\sigma)(\psi^R_m\frac{e^{2\sigma}}{\sqrt{r}}\hat{f}^{R*}_m) \tag{B.8}$$

$$= -\sum_{m,n}\frac{1}{2r}\bar{\psi}^L_n\left\{e^{2\sigma}\hat{f}^{L*}_n(e^{-4\sigma}\partial_\phi+\partial_\phi e^-4\sigma)e^{2\sigma}\hat{f}^{R*}_m\right\}\psi^R_m \tag{B.9}$$

$$= -\sum_{m,n}\frac{1}{2r}\bar{\psi}^L_n e^{2\sigma}\hat{f}^{L*}_n\left\{e^{-4\sigma}(2\frac{\partial\sigma}{\partial\phi}+\hat{f}'^R_m)+(-\frac{\partial\sigma}{\partial\phi}+\hat{f}'^R_m)\right\}e^{2\sigma}\psi^R_m \tag{B.10}$$

$$= -\sum_{m,n}\bar{\psi}^L_n\psi^R_m\;\hat{f}^{L*}_n(\frac{1}{r}\partial_\phi)\hat{f}^R_m. \tag{B.11}$$

Note that corresponding term with $L\leftrightarrow R$ has a relative sign. Moving on to the bulk mass term,

$$-m\;re^{-4\sigma}\bar{\Psi}_L\Psi_R = -\sum_{m,n}m\;re^{-4\sigma}\left(\bar{\psi}^L_n\frac{e^{2\sigma}}{\sqrt{r}}\hat{f}^{L*}_n\right)\left(\psi^R_m\frac{e^{2\sigma}}{\sqrt{r}}\hat{f}^R_m\right) \tag{B.12}$$

$$= -m\;\bar{\psi}^L_n\psi^R_m\;\hat{f}^{L*}_n\hat{f}^R_m. \tag{B.13}$$

Combining (B.11) and (B.13) and setting equal to the 4D mass term in the effective theory, we have

$$\begin{aligned}-\sum_n m_n\;\bar{\psi}_n\psi_n = -\sum_{m,n}\int d\phi\;\Bigg\{&\left[\bar{\psi}^L_n\psi^R_m\;\hat{f}^{L*}_n(\frac{1}{r}\partial_\phi)\hat{f}^R_m-(L\leftrightarrow R)\right]\\ &-\left[m\bar{\psi}^L_n\psi^R_m\;\hat{f}^{L*}_n\hat{f}^R_m+(L\leftrightarrow R)\right]\Bigg\}\end{aligned} \tag{B.14}$$

By using the orthogonality relation (B.7) and combining chiral spinors into Dirac spinors, we finally get the desired differential equation (4.8) for the $\hat{f}$'s,

$$\left(\pm\frac{1}{r}\partial_\phi-m\right)\hat{f}^{L,R}_n(\phi) = -m_n e^{\sigma}\hat{f}^{R,L}_n(\phi). \tag{B.15}$$

C. Highlights of the calculation $\mu\to e\gamma$ in RS1 with bulk neutrinos

Here we derive the result (5.1) for the radiative decay $\mu\to e\gamma$. The result is given by

$$\mathcal{M} = \sum_i\mathcal{M}(m_i), \tag{C.1}$$

where $\mathcal{M}(m_i)$ is the amplitude mediated by a neutrino of mass m_i. It is sufficient, then, to calculate a general $\mathcal{M}(m_i)$, which we will henceforth refer to as $\mathcal{M}$ to simplify notation. We provide some details following the texts by Cheng and Li [43,44]. I have chosen to highlight some details of the calculation not made explicit in the texts.

C.1 Lorentz structure

Let us begin with the Lorentz structure of the amplitude, $\mathcal{M} = \epsilon^\lambda\langle e|J^{\rm EM}_\lambda|\mu\rangle$. We can expand the bra-ket into the following basis of 4×4 matrices,

$$\langle e|J^{\rm EM}_\lambda|\mu\rangle = \bar{u}_e(p-q)\Big\{iq^\nu\sigma_{\lambda\nu}(A+B\gamma_5)+\gamma_\lambda(C+D\gamma_5)+q_\lambda(E+F\gamma_5)\Big\}u_\mu(p). \tag{C.2}$$

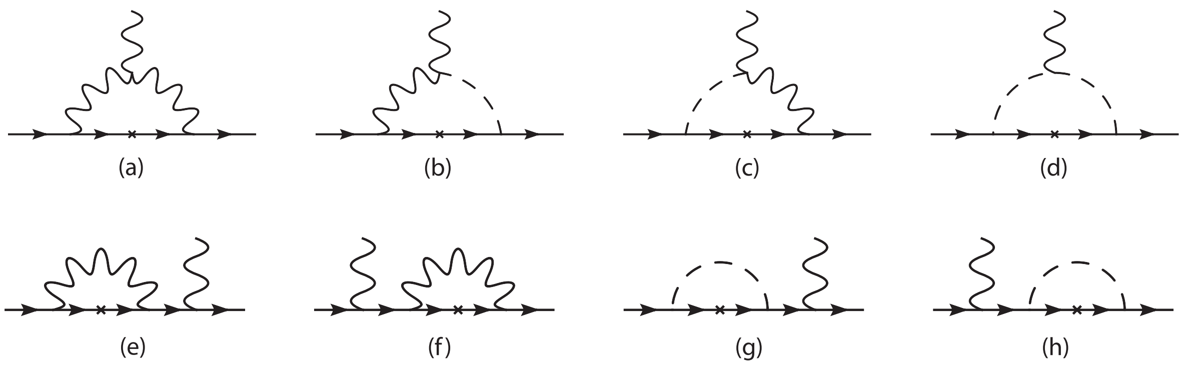

Figure 6: All one-loop diagrams that must be considered for the process $\mu \to e\gamma$. Internal gauge particles are W or Higgs bosons. Diagrams (e)-(h) cannot contribute due to their Lorentz structure.

We can drop the last term in the brackets because $\epsilon \cdot q = 0$ so it makes no contribution to $\mathcal{M}(m_i)$ on-shell. We now invoke the Ward identity, $q^\lambda \langle e|J^{\text{EM}}_\lambda|\mu\rangle = 0$.

$$q^\lambda \langle e|J^{\text{EM}}_\lambda|\mu\rangle = \langle e|\left\{ iq^\lambda q^\nu \sigma_{\lambda\nu}(A + B\gamma_5) + \not{q}(C + D\gamma_5)\right\}|\mu\rangle \tag{C.3}$$

$$= \bar{u}_e(p-q)\left\{ -m_e(C + D\gamma_5) + m_\mu(C - D\gamma_5)\right\}u_\mu(p). \tag{C.4}$$

We have made use of the fact that the $(A + B\gamma_5)$ term vanishes by the antisymmetry of $\sigma_{\lambda\nu}$ and that the remaining term can be simplified using $q = p + (q - p)$ and the Dirac equation $(\not{p} - m)u(p) = 0$ to swap the $\not{q}$ with m_e and m_μ. Because the resulting expression must vanish, $C = D = 0$.

Since $m_e \ll m_\mu$, we can effectively set $m_e = 0$. In this case the left and right handed components decouple, hence $A = \pm B$ so that $(A + B\gamma_5) = A(1 \pm \gamma_5)$ becomes a projection operator. Since this process only occurs for left-handed electrons, we pick $A = B$ and thus our amplitude takes the form of a magnetic transition,

$$\mathcal{M} = A\bar{u}_e(p-q)i\sigma_{\lambda\nu}q^\nu(1 + \gamma_5)u_\mu(p) \tag{C.5}$$

$$= A\bar{u}_e(p-q)(1 + \gamma_5)(2p \cdot \epsilon - m_\mu \gamma \cdot \epsilon)u_\mu(p). \tag{C.6}$$

In the second line we have made use of the Gordon decomposition, which follows from the anticommutation relations of the γ-matrices and the Dirac equation. A is called the **invariant amplitude**. Knowing this form for $\mathcal{M}$ allows us to simplify the evaluation of our diagrams. Since we know the full structure of the final result, we only need to constrain the constant A. Thus we only need to calculate the component of each diagram that is proportional to $(p \cdot \epsilon)\ \bar{u}(1 + \gamma_5)u$. All other components of contributing diagrams must cancel. In particular, the loops in the last four diagrams in Figure 6 only contribute to the flavor mixing angle. These diagrams thus make no net contribution to the final amplitude since they are proportional to $\epsilon_\mu \bar{u}\gamma^\mu u$ rather than $\bar{u}(1 + \gamma_5)u$ as we require.

C.2 Details of $\mu \to W\nu\gamma \to e\gamma$

Let us now demonstrate the machinery for calculating the amplitude $\mathcal{M}_a$ of diagram (a) in Figure 6. The generalization to the other contributing diagrams is straightforward and we will only mention the

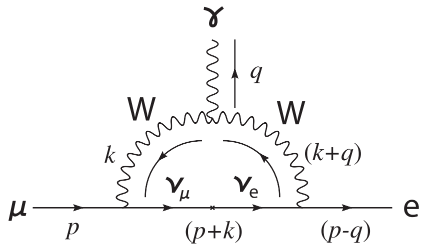

Figure 7: The assignment of momenta for the calculation of $\mu \to e\gamma$.

results of those analogous calculations. The assignment of momentum variables is shown in Figure 7. Applying the Feynman rules in 't Hooft-Feynman gauge,

$$\mathcal{M}_a = -i\sum_i \int \frac{d^4x}{(2\pi)^4} \bar{u}_e(p-q)\left(\frac{ig}{2\sqrt{2}}\right) U^*_{ei}\gamma_\mu(1-\gamma_5)\frac{i}{(\not{p}+\not{k})-m_i}\left(\frac{ig}{2\sqrt{2}}\right)$$
$$\times U_{\mu i}\gamma_\nu(1-\gamma_5)u_\mu(p)\frac{-ig^{\nu\beta}}{k^2-M_W^2}\frac{-ig^{\mu\alpha}}{(k+q)^2-M_W^2}(-ie\Gamma_{\alpha\beta}) \tag{C.7}$$

$$\Gamma_{\alpha\beta} = (2k\cdot\epsilon)g_{\alpha\beta} - (k+2q)_\beta\epsilon_\alpha - (k-q)_\nu\epsilon_\beta \tag{C.8}$$

Where Γ contains the $WW\gamma$ vertex and the photon polarization. We can simplify this by writing the spin contraction as $N^{\mu\nu}$,

$$\mathcal{M}_a = -i\frac{g^2e}{4}\sum_i U^*_{ei}U_{\mu i}\int \frac{d^4k}{(2\pi)^4}\frac{N^{\mu\nu}\Gamma_{\mu\nu}}{[(p+k)^2-m_i^2][k^2-M_W^2][(k+q)^2-M_W^2]} \tag{C.9}$$

$$N_{\mu\nu} = \bar{u}_e(p-q)\gamma_\mu(\not{p}+\not{k})\gamma_\nu(1-\gamma_5)u_\mu(p). \tag{C.10}$$

The natural step is to introduce Feynman parameters to combine the denominators. For our case the relevant expression is

$$\frac{1}{A_1}\frac{1}{A_2}\frac{1}{A_3} = 2!\int_0^1 \frac{d\alpha_1\, d\alpha_2\, \delta(1-\alpha_1-\alpha_2)}{[\alpha_1A_1+\alpha_2A_2+(1-\alpha_1-\alpha_2)A_3]^3} \tag{C.11}$$

$$\tag{C.12}$$

where the A_i are given by the denominators in (C.9). One finds that

$$\alpha_1A_1+\alpha_2A_2+(1-\alpha_1\alpha_2)A_3 = (k+\alpha_1p+\alpha_2q)^2-(1-\alpha_1)M_W^2-\alpha_1m_i^2), \tag{C.13}$$

and hence the denominator becomes 'nice' with respect to the momentum integral if we make the shift $k = \ell+\alpha_1p+\alpha_2q$. Now recall that our goal is to determine the components of this amplitude proportional to $(p\cdot\epsilon)\ \bar{u}(1+\gamma_5)u$. Inserting this shift into the numerator $N^{\mu\nu}\Gamma_{\mu\nu}$ we can isolate the terms proportional to $p\cdot\epsilon$ to get

$$N^{\mu\nu}\Gamma_{\mu\nu} = (p\cdot\epsilon)[\bar{u}(1+\gamma_5)u_\mu]\ 2m_\mu[2(1-\alpha_1)^2+(2\alpha_1-1)\alpha_2]+\cdots. \tag{C.14}$$

Dropping the other terms (which end up canceling, as argued above), we can now perform the momentum integration

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-a^2)^3} = \frac{i}{32\pi^2a^2}, \tag{C.15}$$

where $a^2 = (1-\alpha_1)M_W^2 + \alpha_1 m_i^2$. Plugging all of this into (C.9) and reading off the 'invariant amplitude' A from (C.6), we have

$$A_a = g^2 e \sum_i U_{ei}^* U_{\mu i} (p\cdot\epsilon)[\bar{u}(1+\gamma_5)u_\mu] \int_0^1 d\alpha_1\, d\alpha_2\, \frac{m_\mu[2(1-\alpha_1)^2 + (2\alpha_1 - 1)\alpha_2]}{32\pi^2\,[(1-\alpha_1)M_W^2 + \alpha_1 m_i^2]} \tag{C.16}$$

$$= \sum_i c_i \frac{m_\mu}{16\pi^2} \frac{1}{M_W^2} \int_0^1 d\alpha_1\, \frac{(1-\alpha_1)^2(\frac{3}{2} - \alpha_1)}{[(1-\alpha_1) + \alpha_1(m_i^2/M_W^2)]}, \tag{C.17}$$

where we've written $c_i = \frac{1}{4} g^2 e U_{ei}^* U_{\mu i}$.

C.3 Results for the remaining diagrams

One can now perform the calculations for the remaining diagrams. The machinery is completely analogous: combine the denominator with Feynman parameters, shift the momentum variable, and then isolate the terms proportional to $(p\cdot\epsilon)$ in the numerator. One can then perform the momentum and α_2 integral for these terms. Summing these all together one gets the result (5.1),

$$\mathcal{M}(m_i) = \frac{m_\mu}{M_W^2} \sum_i c_i\; F\!\left(\frac{m_i^2}{M_W^2}\right)\; \bar{u}_e(p-q) i\sigma_{\lambda\nu} q^\nu (1+\gamma_5) u_\mu(p) \tag{C.18}$$

$$F(z) = \int_0^1 d\alpha\, \frac{(1-\alpha)}{(1-\alpha) + \alpha z}[2(1-\alpha)(2-\alpha) + \alpha(1+\alpha)z]. \tag{C.19}$$

The expression for $F(z)$ can be integrated explicitly. This is done in [44] or easily in Mathematica (there's no physics to be extracted from this calculation). One ends up with the explicit form in (5.2): $F(z) = \left(10 - 43z + 78z^2 - 49z^3 - 18z^3 \ln z + 4z^4\right)/(12(1-z)^4)$.

One might be concerned that the other diagrams differ by replacing one or both W propagators with Higgs propagators, and hence it's not clear that the terms in the invariant amplitudes are all proportional to $e\, g^2 m_\mu / M_W^2$. However, one can write out the Higgs propagator and couplings to various particles in terms of M_W and we end up with the correct factors as one might be able to guess from dimensional arguments.

References

[1] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, *The hierarchy problem and new dimensions at a millimeter*, *Phys. Lett.* **B429** (1998) 263–272, [hep-ph/9803315].

[2] L. Randall and R. Sundrum, *A large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370–3373, [hep-ph/9905221].

[3] K. Akama, *An early proposal of 'brane world'*, *Lect. Notes Phys.* **176** (1982) 267–271, [hep-th/0001113].

[4] V. A. Rubakov and M. E. Shaposhnikov, *Do we live inside a domain wall?*, *Phys. Lett.* **B125** (1983) 136–138.

[5] **Super-Kamiokande** Collaboration, Y. Ashie *et al.*, *Evidence for an oscillatory signature in atmospheric neutrino oscillation*, *Phys. Rev. Lett.* **93** (2004) 101801, [hep-ex/0404034].

[6] M. Jezabek and Y. Sumino, *Neutrino mixing and seesaw mechanism*, *Phys. Lett.* **B440** (1998) 327–331, [hep-ph/9807310].

[7] N. Arkani-Hamed, S. Dimopoulos, G. R. Dvali, and J. March-Russell, *Neutrino masses from large extra dimensions*, *Phys. Rev.* **D65** (2002) 024032, [hep-ph/9811448].

[8] Y. Grossman and M. Neubert, *Neutrino masses and mixings in non-factorizable geometry*, *Phys. Lett.* **B474** (2000) 361–371, [hep-ph/9912408].

[9] R. Kitano, *Lepton flavor violation in the randall-sundrum model with bulk neutrinos*, *Phys. Lett.* **B481** (2000) 39–44, [hep-ph/0002279].

[10] T. P. Cheng and L.-F. Li, *Heavy particle electroweak loop effects in extra- dimensional models with bulk neutrinos*, *Phys. Lett.* **B502** (2001) 152–158, [hep-ph/0101068].

[11] R. Sundrum, *Effective field theory for a three-brane universe*, *Phys. Rev.* **D59** (1999) 085009, [hep-ph/9805471].

[12] P. Horava and E. Witten, *Heterotic and type i string dynamics from eleven dimensions*, *Nucl. Phys.* **B460** (1996) 506–524, [hep-th/9510209].

[13] E. Witten, *Strong coupling expansion of calabi-yau compactification*, *Nucl. Phys.* **B471** (1996) 135–158, [hep-th/9602070].

[14] R. Rattazzi, *Cargese lectures on extra dimensions*, hep-ph/0607055.

[15] H. Haber and A. E. Nelson, *Particle Physics And Cosmology: The Quest For Physics Beyond The Standard Model(s): TASI 2002*. World Scientific, USA (2004).

[16] R. Sundrum, *To the fifth dimension and back. (tasi 2004)*, hep-th/0508134.

[17] S. P. Robinson, *Normalization conventions for newton's constant and the planck scale in arbitrary spacetime dimension*, gr-qc/0609060.

[18] C. Csaki, *Tasi lectures on extra dimensions and branes*, hep-ph/0404096.

[19] R. Wald, *General Relativity*. Chicago University Press. (1984).

[20] W. D. Goldberger and M. B. Wise, *Modulus stabilization with bulk fields*, *Phys. Rev. Lett.* **83** (1999) 4922–4925, [hep-ph/9907447].

[21] J. M. Maldacena, *Lectures on ads/cft*, hep-th/0309246.

[22] N. Arkani-Hamed, M. Porrati, and L. Randall, *Holography and phenomenology*, *JHEP* **08** (2001) 017, [hep-th/0012148].

[23] R. Rattazzi and A. Zaffaroni, *Comments on the holographic picture of the randall-sundrum model*, *JHEP* **04** (2001) 021, [hep-th/0012248].

[24] H. Davoudiasl, J. L. Hewett, and T. G. Rizzo, *Phenomenology of the randall-sundrum gauge hierarchy model*, *Phys. Rev. Lett.* **84** (2000) 2080, [hep-ph/9909255].

[25] R. A. Bertlmann, *Anomalies in Quantum Field Theory*. Oxford, UK: Oxford University Press (1996).

[26] H. Murayama, *Notes on clifford algebra and spin(n) representations*, . Lecture notes from Physics 230A (Spring 2007), University of California Berkeley. http://hitoshi.berkeley.edu/230A/clifford.pdf.

[27] A. N. Redlich, *Gauge noninvariance and parity nonconservation of three- dimensional fermions*, *Phys. Rev. Lett.* **52** (1984) 18.

[28] J. Callan, Curtis G. and J. A. Harvey, *Anomalies and fermion zero modes on strings and domain walls*, *Nucl. Phys.* **B250** (1985) 427.

[29] S. P. Mikheev and A. Y. Smirnov, *Resonance enhancement of oscillations in matter and solar neutrino spectroscopy*, *Sov. J. Nucl. Phys.* **42** (1985) 913–917.

[30] L. Wolfenstein, *Neutrino oscillations in matter*, *Phys. Rev.* **D17** (1978) 2369.

[31] **SNO** Collaboration, Q. R. Ahmad *et al.*, *Measurement of day and night neutrino energy spectra at sno and constraints on neutrino mixing parameters*, *Phys. Rev. Lett.* **89** (2002) 011302, [nucl-ex/0204009].

[32] P. C. de Holanda and A. Y. Smirnov, *Lma msw solution of the solar neutrino problem and first kamland results*, *JCAP* **0302** (2003) 001, [hep-ph/0212270].

[33] W.-M. Yao, C. Amsler, D. Asner, R. Barnett, J. Beringer, P. Burchat, C. Carone, C. Caso, O. Dahl, G. D'Ambrosio, A. DeGouvea, M. Doser, S. Eidelman, J. Feng, T. Gherghetta, M. Goodman, C. Grab, D. Groom, A. Gurtu, K. Hagiwara, K. Hayes, J. Hernández-Rey, K. Hikasa, H. Jawahery, C. Kolda, K. Y., M. Mangano, A. Manohar, A. Masoni, R. Miquel, K. Mönig, H. Murayama, K. Nakamura, S. Navas, K. Olive, L. Pape, C. Patrignani, A. Piepke, G. Punzi, G. Raffelt, J. Smith, M. Tanabashi, J. Terning, N. Törnqvist, T. Trippe, P. Vogel, T. Watari, C. Wohl, R. Workman, P. Zyla, B. Armstrong, G. Harper, V. Lugovsky, P. Schaffner, M. Artuso, K. Babu, H. Band, E. Barberio, M. Battaglia, H. Bichsel, O. Biebel, P. Bloch, E. Blucher, R. Cahn, D. Casper, A. Cattai, A. Ceccucci, D. Chakraborty, R. Chivukula, G. Cowan, T. Damour, T. DeGrand, K. Desler, M. Dobbs, M. Drees, A. Edwards, D. Edwards, V. Elvira, J. Erler, V. Ezhela, W. Fetscher, B. Fields, B. Foster, D. Froidevaux, T. Gaisser, L. Garren, H.-J. Gerber, G. Gerbier, L. Gibbons, F. Gilman, G. Giudice, A. Gritsan, M. Grünewald, H. Haber, C. Hagmann, I. Hinchliffe, A. Höcker, P. Igo-Kemenes, J. Jackson, K. Johnson, D. Karlen, B. Kayser, D. Kirkby, S. Klein, K. Kleinknecht, I. Knowles, R. Kowalewski, P. Kreitz, B. Krusche, Y. Kuyanov, O. Lahav, P. Langacker, A. Liddle, Z. Ligeti, T. Liss, L. Littenberg, L. Liu, K. Lugovsky, S. Lugovsky, T. Mannel, D. Manley, W. Marciano, A. Martin, D. Milstead, M. Narain, P. Nason, Y. Nir, J. Peacock, S. Prell, A. Quadt, S. Raby, B. Ratcliff, E. Razuvaev, B. Renk, P. Richardson, S. Roesler, G. Rolandi, M. Ronan, L. Rosenberg, C. Sachrajda, S. Sarkar, M. Schmitt, O. Schneider, D. Scott, T. Sjöstrand, G. Smoot, P. Sokolsky, S. Spanier, H. Spieler, A. Stahl, T. Stanev, R. Streitmatter, T. Sumiyoshi, N. Tkachenko, G. Trilling, G. Valencia, K. van Bibber, M. Vincter, D. Ward, B. Webber, J. Wells, M. Whalley, L. Wolfenstein, J. Womersley, C. Woody, A. Yamamoto, O. Zenin, J. Zhang, and R.-Y. Zhu, *Review of Particle Physics*, *Journal of Physics G* **33** (2006) 1+.

[34] A. Sirlin, *Ten years of precision electroweak physics*, `hep-ph/9912227`.

[35] **SNO** Collaboration, B. Aharmim *et al.*, *Electron energy spectra, fluxes, and day-night asymmetries of b-8 solar neutrinos from the 391-day salt phase sno data set*, *Phys. Rev.* **C72** (2005) 055502, [`nucl-ex/0502021`].

[36] **Super-Kamiokande** Collaboration, Y. Ashie *et al.*, *Evidence for an oscillatory signature in atmospheric neutrino oscillation*, *Phys. Rev. Lett.* **93** (2004) 101801, [`hep-ex/0404034`].

[37] **CHOOZ** Collaboration, M. Apollonio *et al.*, *Limits on neutrino oscillations from the chooz experiment*, *Phys. Lett.* **B466** (1999) 415–430, [`hep-ex/9907037`].

[38] G. L. Fogli, E. Lisi, A. Marrone, and G. Scioscia, *Super-kamiokande atmospheric neutrino data, zenith distributions, and three-flavor oscillations*, *Phys. Rev.* **D59** (1999) 033001, [`hep-ph/9808205`].

[39] **Chooz** Collaboration, C. Bemporad, *Results from chooz*, *Nucl. Phys. Proc. Suppl.* **77** (1999) 159–165.

[40] A. Strumia and F. Vissani, *Implications of neutrino data circa 2005*, *Nucl. Phys.* **B726** (2005) 294–316, [`hep-ph/0503246`].

[41] S. L. Glashow, J. Iliopoulos, and L. Maiani, *Weak interactions with lepton-hadron symmetry*, *Phys. Rev.* **D2** (1970) 1285–1292.

[42] T. P. Cheng and L.-F. Li, *mu → e gamma in theories with dirac and majorana neutrino mass terms*, *Phys. Rev. Lett.* **45** (1980) 1908.

[43] T.-P. Cheng and L.-F. Li, *Gauge theory of elementary particle physics.* Clarendon Press. (1984) Section 13.3.

[44] T.-P. Cheng and L.-F. Li, *Gauge Theory of Elementary Particle Physics: Problems and Solutions.* Clarendon Press. (2000) Section 13.4.

[45] T. P. Cheng and L.-F. Li, *Effects of superheavy neutrinos in low-energy weak processes*, *Phys. Rev.* **D44** (1991) 1502–1509.

[46] **MEGA** Collaboration, M. L. Brooks *et al.*, *New limit for the family-number non-conserving decay mu+ – ¿ e+ gamma*, *Phys. Rev. Lett.* **83** (1999) 1521–1524, [`hep-ex/9905013`].

[47] **CLEO** Collaboration, S. Ahmed *et al.*, *Update of the search for the neutrinoless decay tau –¿ mu gamma*, *Phys. Rev.* **D61** (2000) 071101, [hep-ex/9910060].

[48] **CLEO** Collaboration, K. W. Edwards *et al.*, *Search for neutrinoless tau decays: tau –¿ e gamma and tau –¿ mu gamma*, *Phys. Rev.* **D55** (1997) 3919–3923.

[49] N. Arkani-Hamed, H.-C. Cheng, J. L. Feng, and L. J. Hall, *Probing lepton flavor violation at future colliders*, *Phys. Rev. Lett.* **77** (Sep, 1996) 1937–1940.

[50] T. Gherghetta and A. Pomarol, *Bulk fields and supersymmetry in a slice of ads*, *Nucl. Phys.* **B586** (2000) 141–162, [hep-ph/0003129].

[51] E. O. Iltan, *Radiative lepton flavor violating decays in the randall sundrum background with localized leptons*, arXiv:0704.4000 [hep-ph].

[52] H.-C. Cheng, K. T. Matchev, and M. Schmaltz, *Bosonic supersymmetry? getting fooled at the lhc*, *Physical Review D* **66** (2002) 056006.

[53] C. Csaki, M. Graesser, C. F. Kolda, and J. Terning, *Cosmology of one extra dimension with localized gravity*, *Phys. Lett.* **B462** (1999) 34–40, [hep-ph/9906513].

[54] K. Agashe, G. Perez, and A. Soni, *Flavor structure of warped extra dimension models*, *Phys. Rev.* **D71** (2005) 016002, [hep-ph/0408134].

[55] E. Ma and G. Rajasekaran, *Softly broken a_4 symmetry for nearly degenerate neutrino masses*, *Physical Review D* **64** (2001) 113012.

[56] **The MiniBooNE** Collaboration, A. A. Aguilar-Arevalo *et al.*, *A search for electron neutrino appearance at the $\delta m^2 \sim 1$ ev^2 scale*, arXiv:0704.1500 [hep-ex].

[57] D. Binosi and L. Theussl, *Jaxodraw: A graphical user interface for drawing feynman diagrams*, *Comput. Phys. Commun.* **161** (2004) 76–86, [hep-ph/0309015].