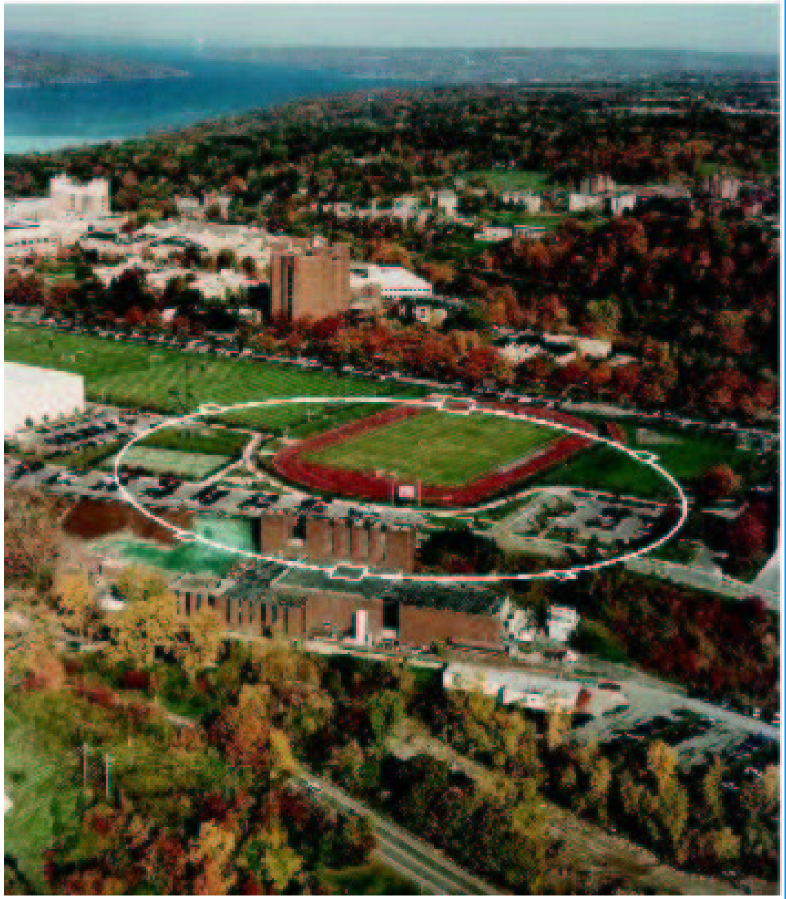
Introduction to Accelerator Physics and Technology

Content



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Literature

Required:
The Physics of Particle Accelerators, Klaus Wille, Oxford University Press, 2000, ISBN: 19 850549 3

Optional:
Particle Accelerator Physics I, Helmut Wiedemann, Springer, 2nd edition, 1999, ISBN 3 540 64671 x

Related material:
Handbook of Accelerator Physics and Engineering, Alexander Wu Chao and Maury Tigner, 2nd edition, 2002, World Scientific, ISBN: 981 02 3858 4

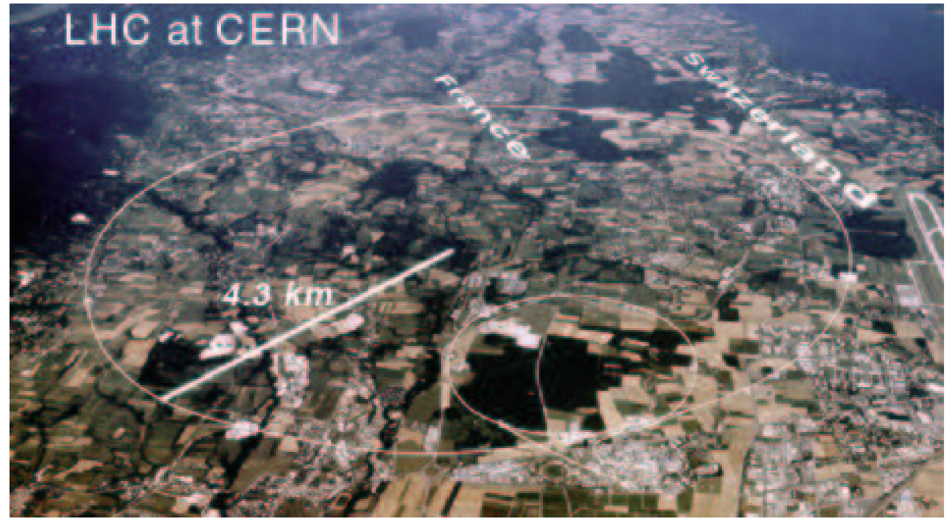
Particle Accelerator Physics II, Helmut Wiedemann, Springer, 2nd edition, 1999, ISBN 3 540 64504 7

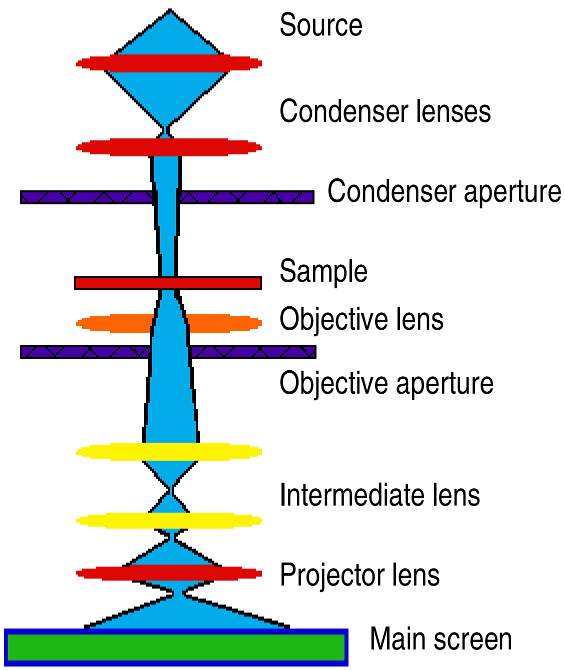
What is accelerator physics

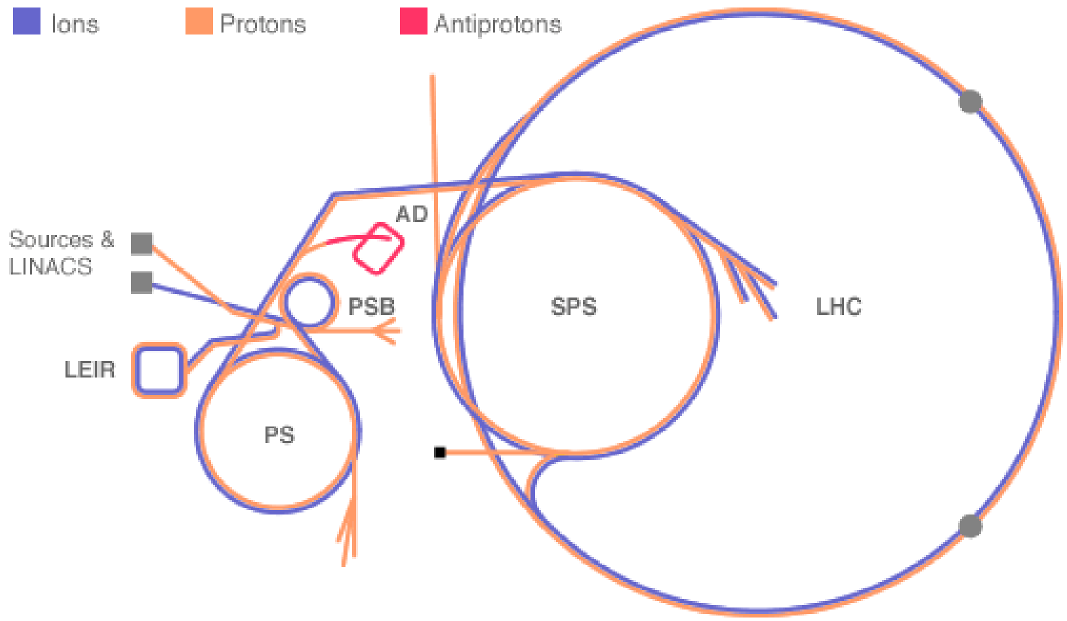
Accelerator Physics has applications in particle accelerators for high energy physics or for x-ray science, in spectrometers, in electron microscopes, and in lithographic devices. These instruments have become so complex that an empirical approach to properties of the particle beams is by no means sufficient and a detailed theoretical understanding is necessary. This course will introduce into theoretical aspects of charged particle beams and into the technology used for their acceleration.

- Physics of beams
- Physics of non-neutral plasmas
- Physics of involved in the technology:
  - Superconductivity in magnets and radiofrequency (RF) devices
  - Surface physics in particle sources, vacuum technology, RF devices
  - Material science in collimators, beam dumps, superconducting materials

Different accelerators







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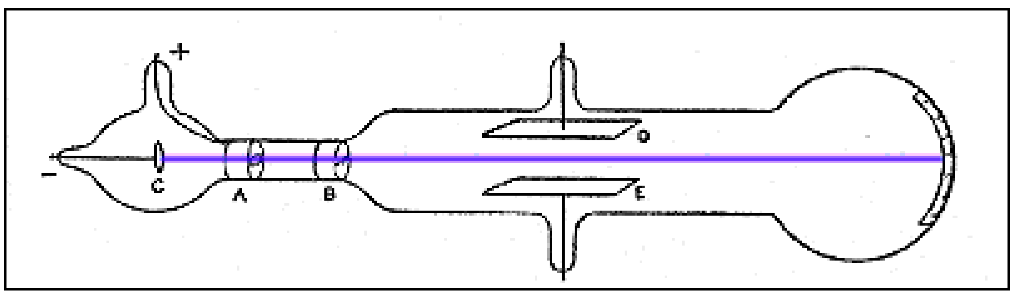
A short history of accelerators

- 1862: Maxwell theory of electromagnetism
- 1887: Hertz discovery of the electromagnetic wave
- 1886: Goldstein discovers positively charged rays (ion beams)
- 1894: Lenard extracts cathode rays (with a 2.65um Al Lenard window)
- 1897: JJ Thomson shows that cathode rays are particles since they followed the classical Lorentz force $m\vec{a} = e(\vec{E} + \vec{v} \times \vec{B})$ in an electromagnetic field
- 1926: GP Thomson shows that the electron is a wave (1929-1930 in Cornell, NP in 1937)

NP 1905

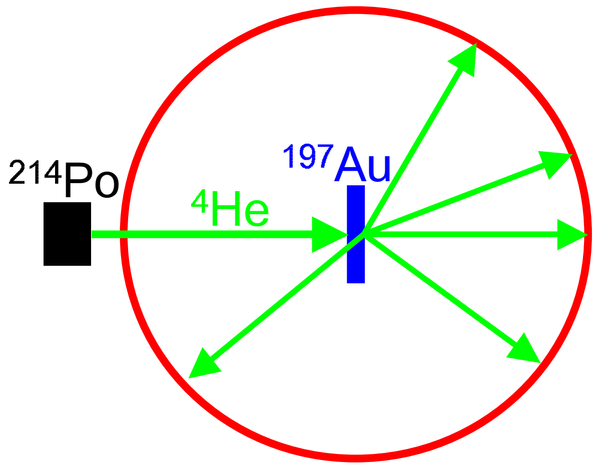
Philipp E.A. von Lenard
Germany 1862-1947



NP 1906
Joseph J. Thomson
UK 1856-1940

A short history of accelerators

- 1895: Roentgen discovers x-rays with cathode rays
- 1911: Rutherford discovers the nucleus with 7.7MeV ^{4}He from ^{214}Po alpha decay measuring the elastic crossection of ^{197}Au + ^{4}He $\mapsto$ ^{197}Au + ^{4}He.



$$E = \frac{Z_1 e Z_2 e}{4\pi\varepsilon_0 d} = Z_1 Z_2 m_e c^2 \frac{r_e}{d},$$

$$r_e = 2.8\mathrm{fm}, \quad m_e c^2 = 0.511\mathrm{MeV}$$

- 1919: Rutherford produces first nuclear reactions with natural ^{4}He
  ^{14}N + ^{4}He $\mapsto$ ^{17}O + p
- 1921: Greinacher invents the cascade generator for several 100 keV
- Rutherford is convinced that several 10 MeV are in general needed for nuclear reactions. He therefore gave up the thought of accelerating particles.

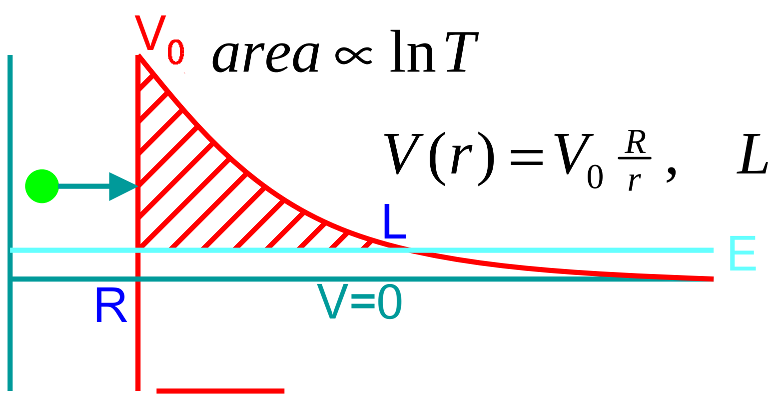
Tunneling allows low energies

- 1928: Explanation of alpha decay by Gamov as tunneling showed that several 100keV protons might suffice for nuclear reactions

Schroedinger equation: $\frac{\partial^2}{\partial r^2}u(r) = \frac{2m}{\hbar^2}[V(r)-E]u(r)\,,\quad T = \left|\frac{u(L)}{u(0)}\right|^2$

The transmission probability T for an alpha particle traveling from the inside towards the potential well that keeps the nucleus together determines the lifetime for alpha decay.

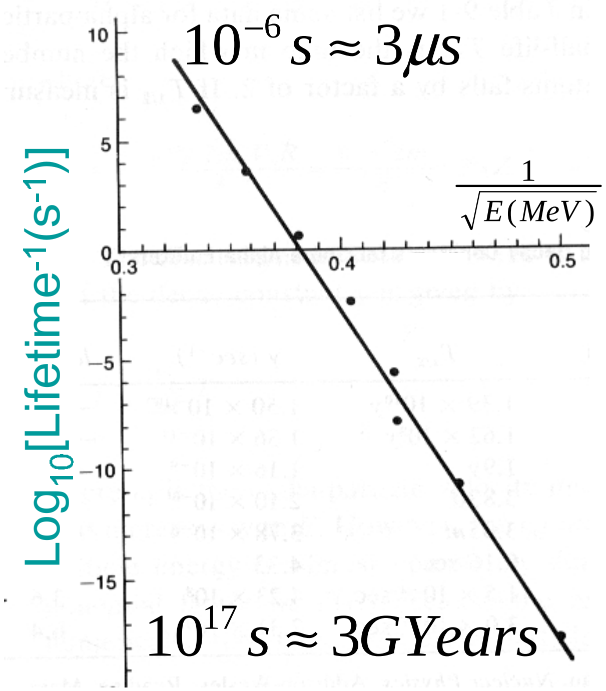
$area \propto \ln T$



$$V(r) = V_0\frac{R}{r}\,,\quad L = \frac{V_0}{E}R$$

$$T \approx \exp[-2\int_R^L \frac{\sqrt{2m[V(r)-E]}}{\hbar}dr]$$

$$\ln T \approx A - \frac{C}{\sqrt{E}}$$



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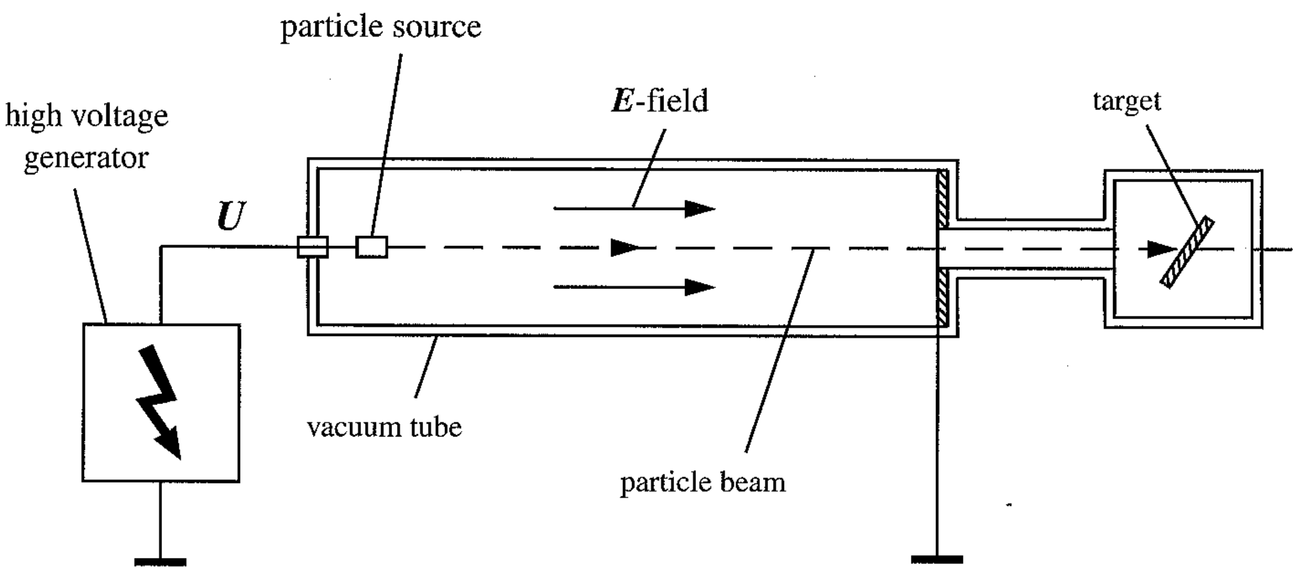
Three historic lines of accelerators

Direct Voltage Accelerators   Resonant Accelerators   Transformer Accelerator



Voltage 1MV
Charge Ze
Energy Z MeV

The energy limit is given by the maximum possible voltage. At the limiting voltage, electrons and ions are accelerated to such large energies that they hit the surface and produce new ions. An avalanche of charge carries causes a large current and therefore a breakdown of the voltage.

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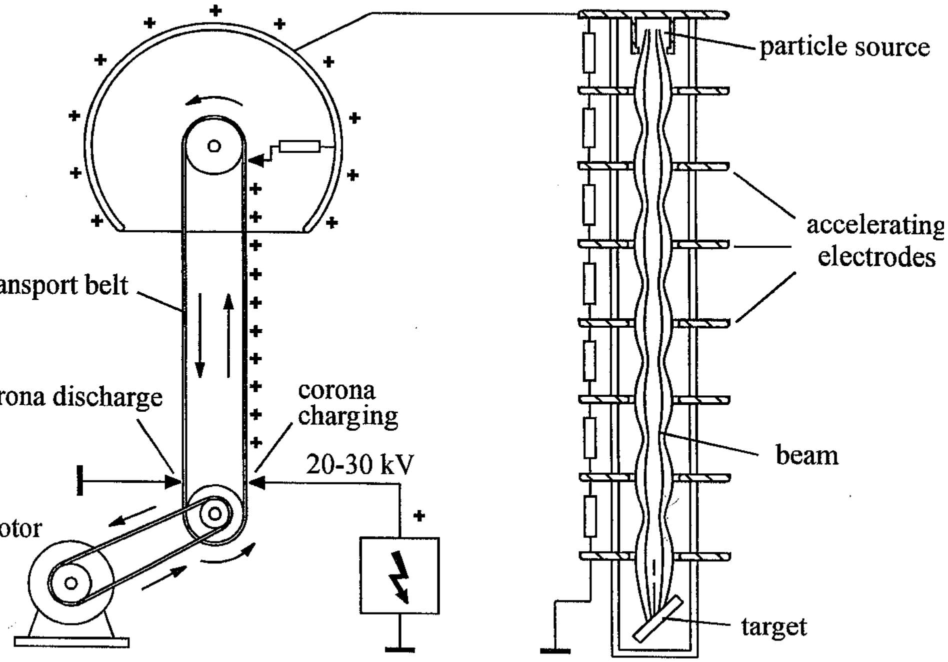
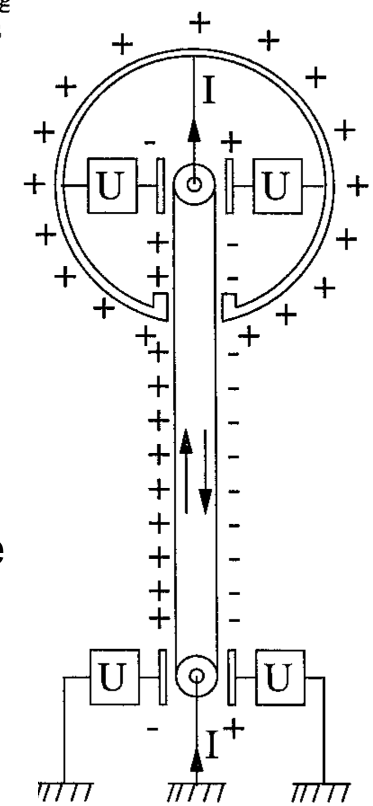
The Van de Graaff Accelerator

- 1930: van de Graaff builds the first 1.5MV high voltage generator





Van de Graaff

- Today Peletrons (with chains) or Laddertron (with stripes) that are charged by influence are commercially available.
- Used as injectors, for electron cooling, for medical and technical n-source via $d + t \mapsto n + \alpha$
- Up to 17.5 MV with insulating gas (1MPa SF_6)

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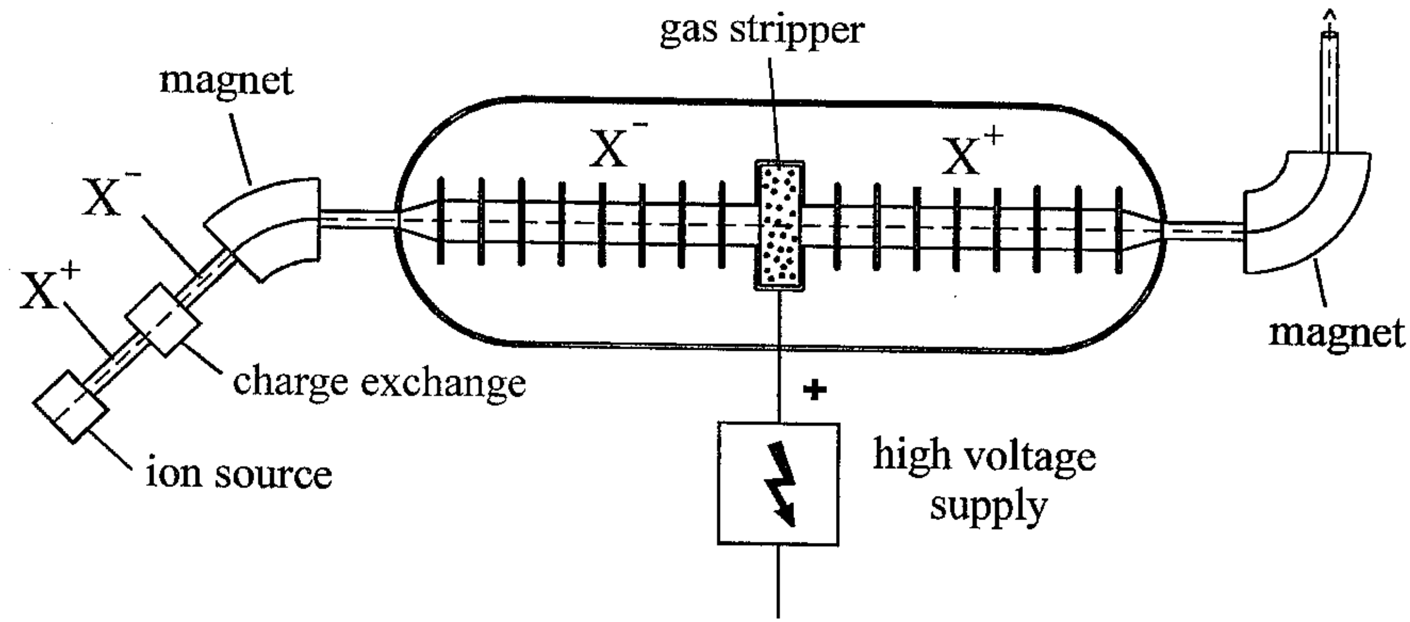
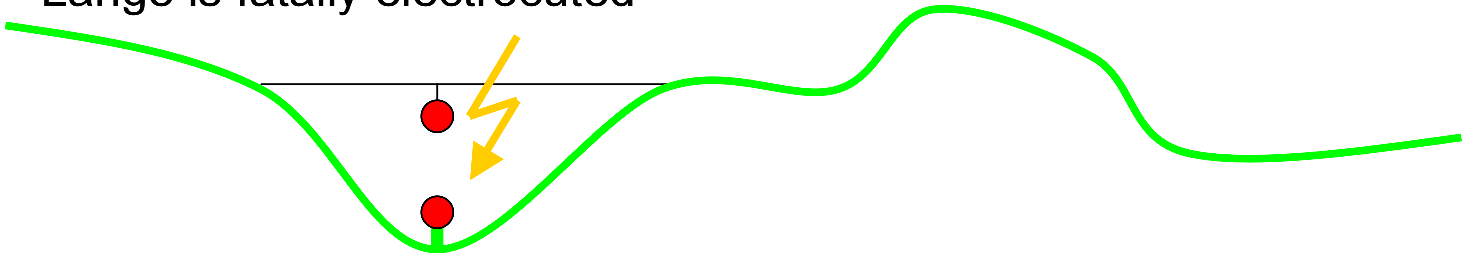
The Tandem Accelerator

- Extension:
  - Two Van de Graaffs, one + one –
  - The Tandem Van de Graaff, highest energy 35MeV

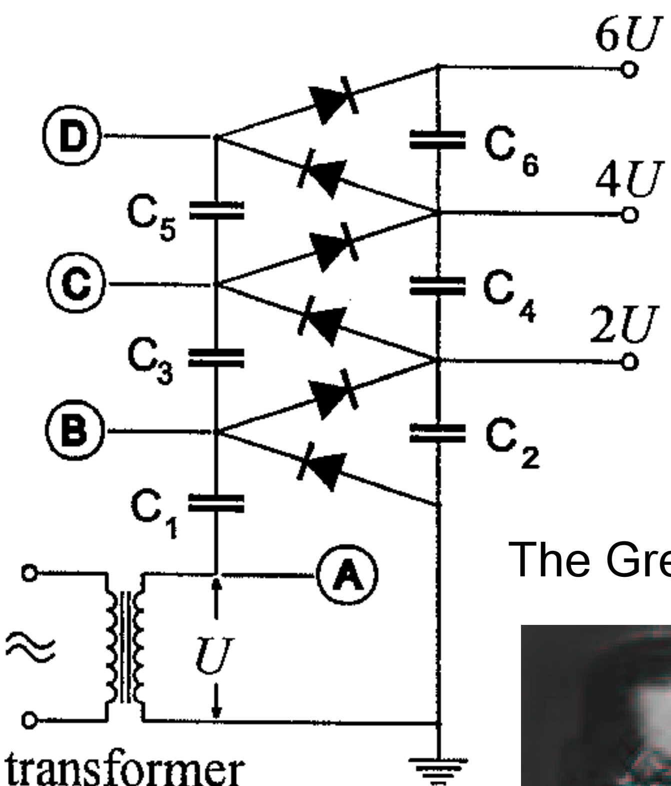


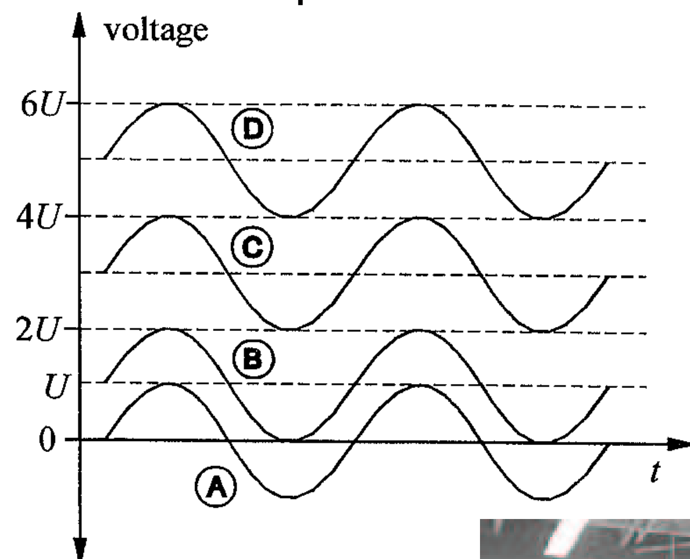
- 1932: Brasch and Lange use potential from lightening, in the Swiss Alps, Lange is fatally electrocuted



The Cockcroft-Walton Accelerator

- 1932: Cockcroft and Walton 1932: 700keV cascate generator (planed for 800keV) and use initially 400keV protons for $^7Li + p \mapsto {}^4He + {}^4He$ and $^7Li + p \mapsto {}^7Be + n$





The Greinacker circuit

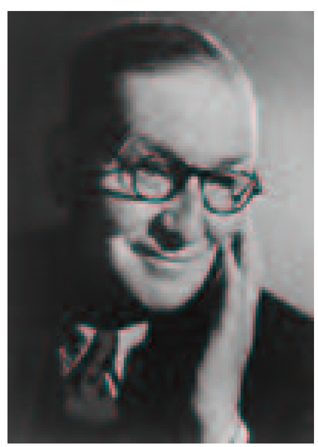
Li

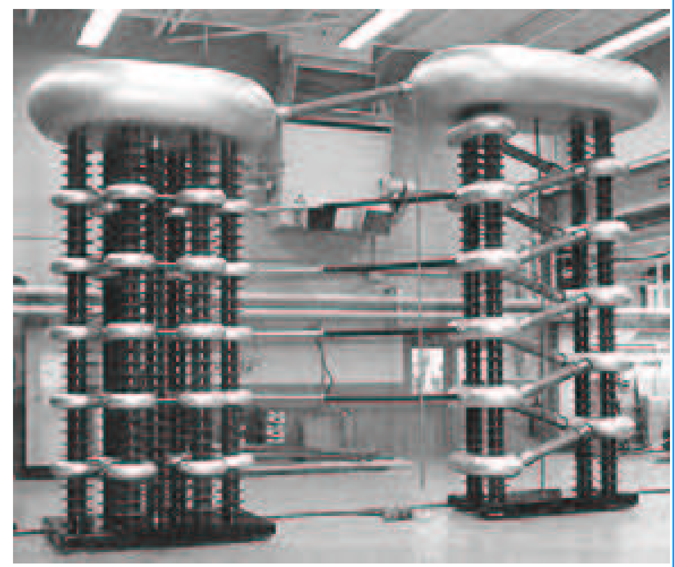
Up to 4MeV, 1A

NP 1951

Sir John D Cockcrof

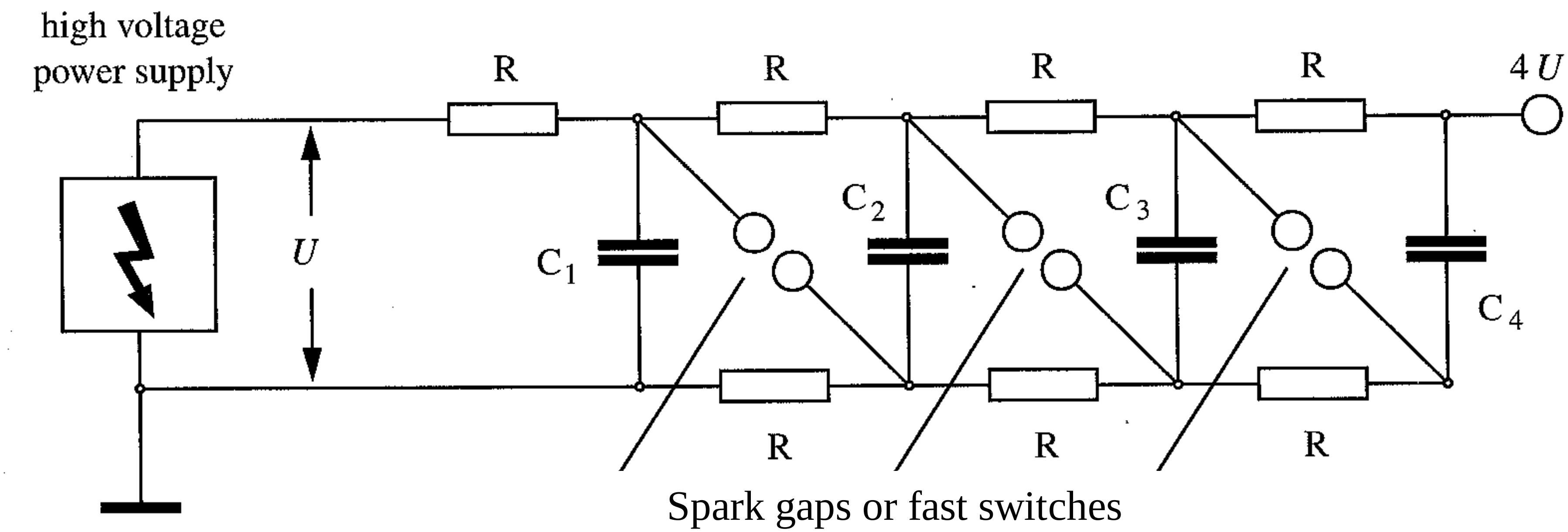
Ernest T S Walton





The Marx Generator

- 1932: Marx Generator achieves 6MV at General Electrics



After capacitors of around 2uF are filled to about 20kV, the spark gaps or switches close as fast as 40ns, allowing up to 500kA.

Today:
The Z-machine (Physics Today July 2003) for z-pinch initial confinement fusion has 40TW for 100ns from 36 Marx generators

Three historic lines of accelerators

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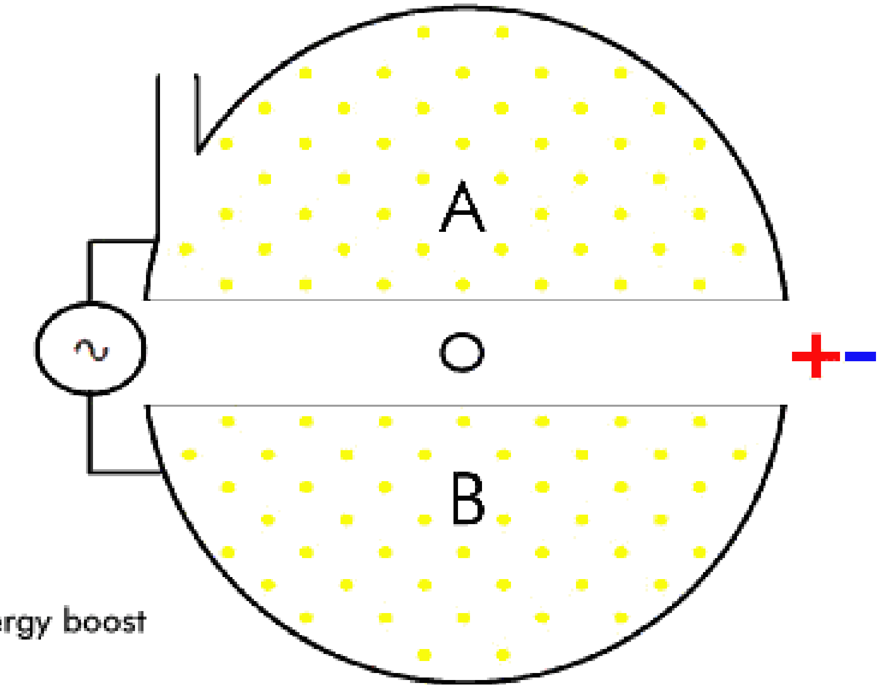
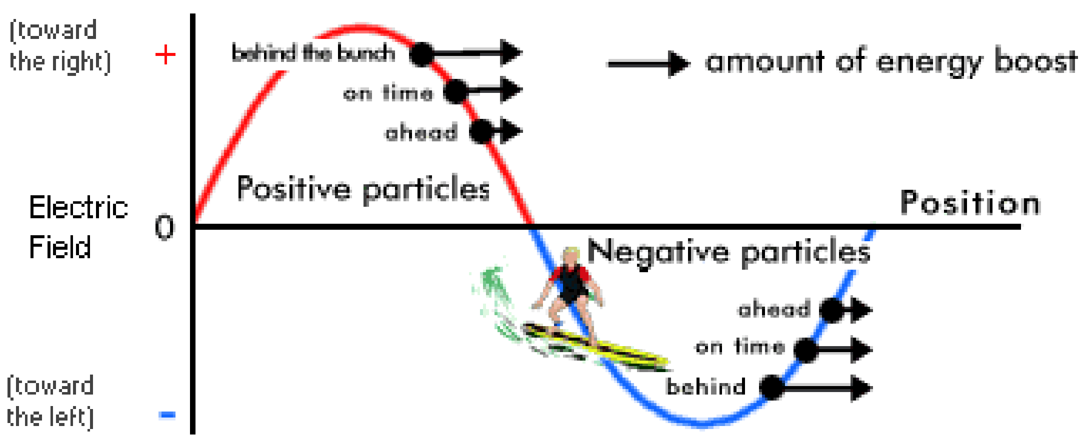
Resonant Accelerators

Direct Voltage Accelerators

Transformer Accelerator





Particles must have the correct phase relation to the accelerating voltage.

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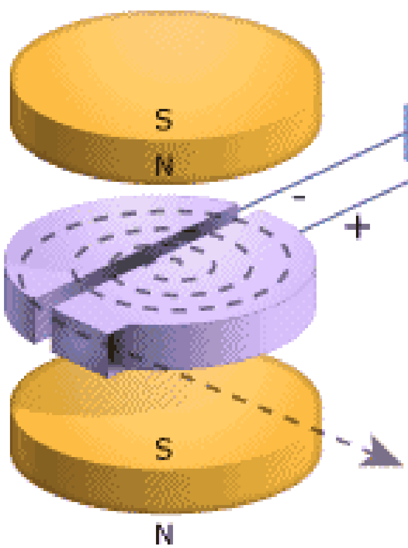
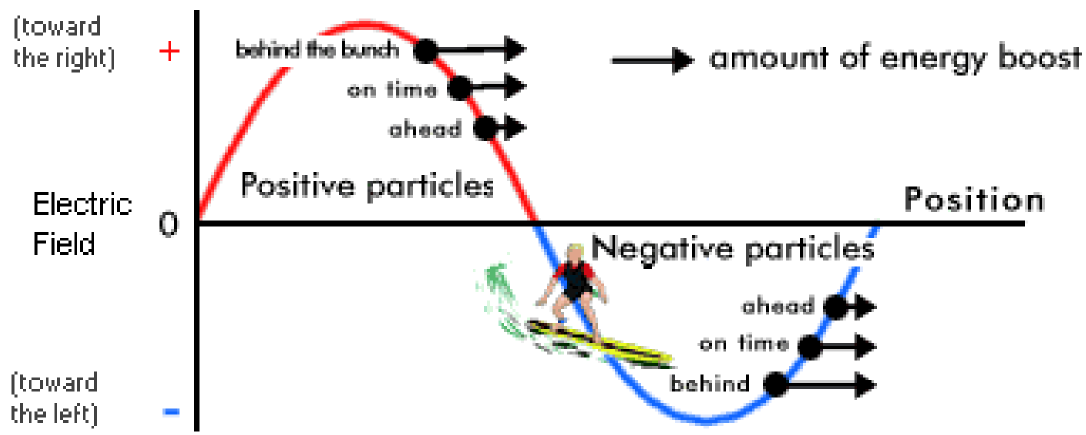
The Cyclotron



- 1930: Lawrence proposes the Cyclotron
  (before he develops a workable color TV screen)
- 1932: Lawrence and Livingston use a cyclotron for
  1.25MeV protons and mention longitudinal
  (phase) focusing



NP 1939
Ernest O Lawrence
USA 1901-1958

- 1934: Livingston builds the first Cyclotron
  away from Berkely (2MeV protons)
  at Cornell (in room B54)

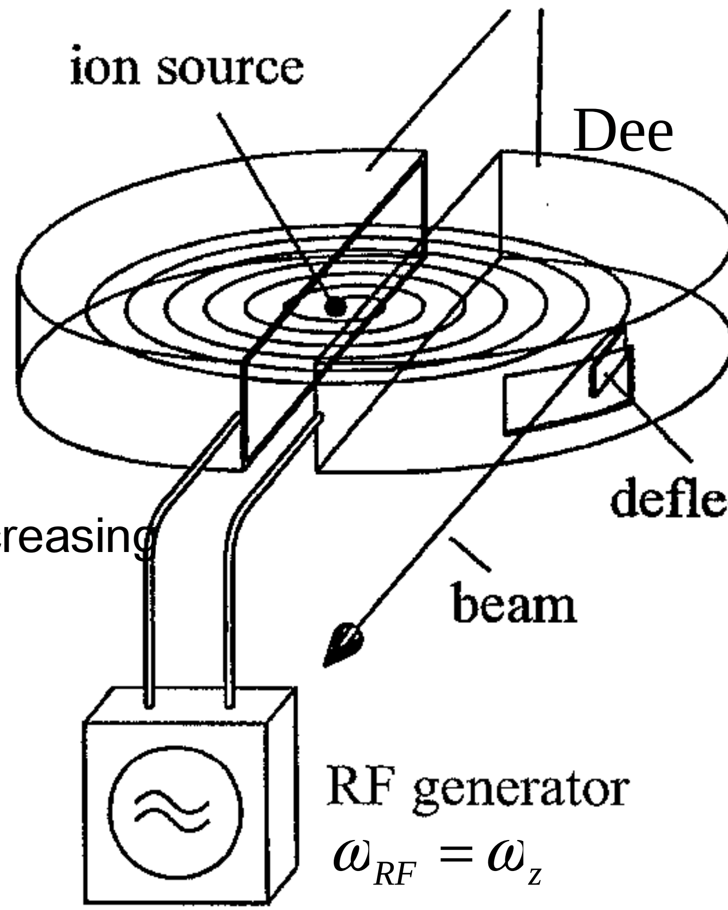
M Stanley Livingston
USA 1905-1986

The cyclotron frequency

$$F_r = m_0 \gamma \omega_z v = q v B_z$$

$$\omega_z = \frac{q}{m_0 \gamma} B_z = \text{const}$$

Condition: Non-relativistic particles.
Therefore not for electrons.



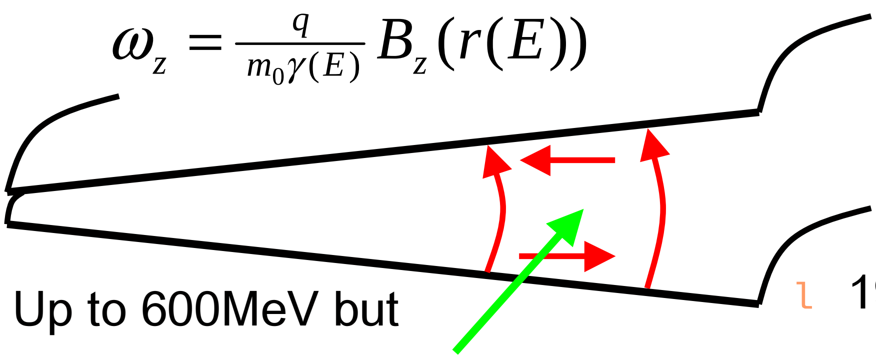
- The synchrocyclotron:

  Acceleration of bunches with decreasing

$$\omega_z(E) = \frac{q}{m_0 \gamma(E)} B_z$$

- The isocyclotron with constant

$$\omega_z = \frac{q}{m_0 \gamma(E)} B_z(r(E))$$



Up to 600MeV but
this vertically defocuses the beam

- 1938: Thomas proposes strong (transverse) focusing for a cyclotron

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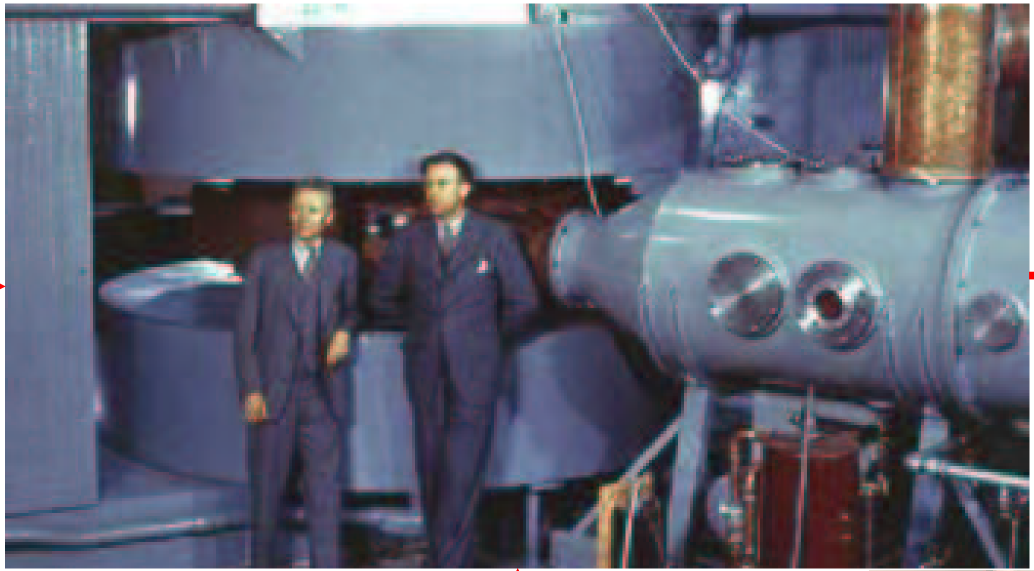
First Medical Applications

- 1939: Lawrence uses 60' cyclotron for 9MeV protons, 19MeV deuterons, and 35MeV 4He. First tests of tumor therapy with neutrons via d + t $\mapsto$ n + α With 200-800keV d to get 10MeV neutrons.



Modern Nuclear Therapy

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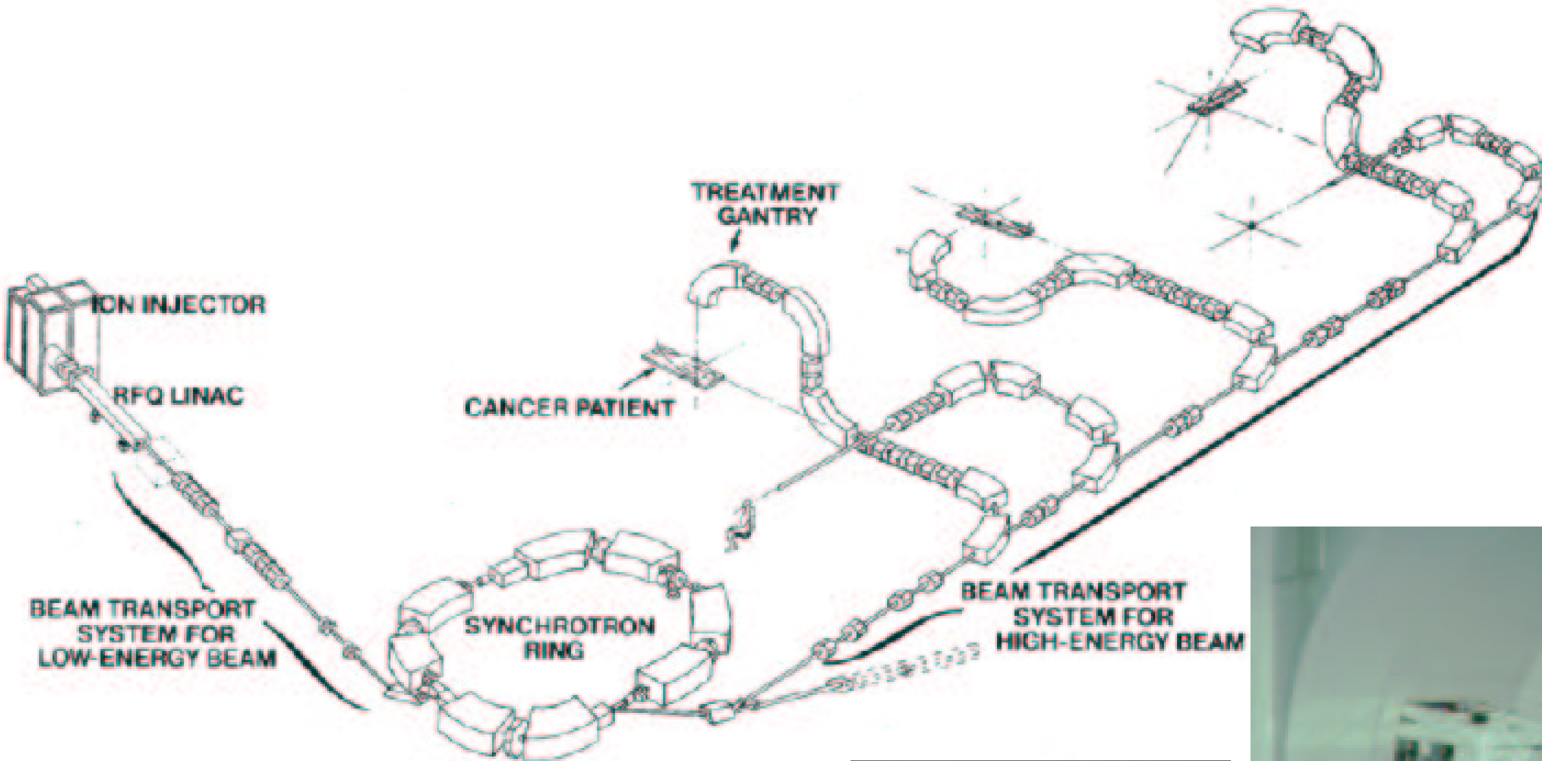
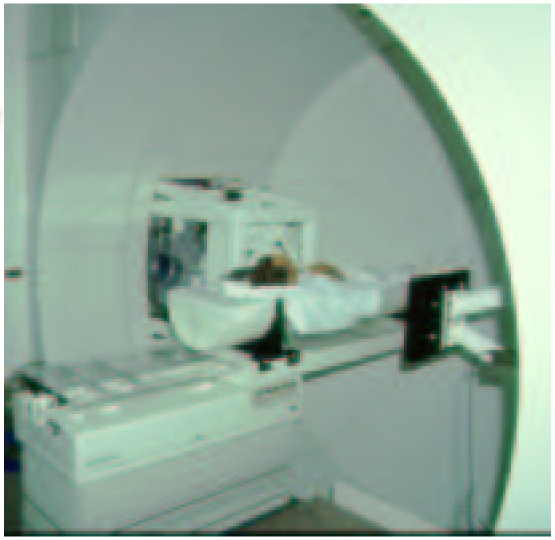
The Loma Linda proton therapy facility





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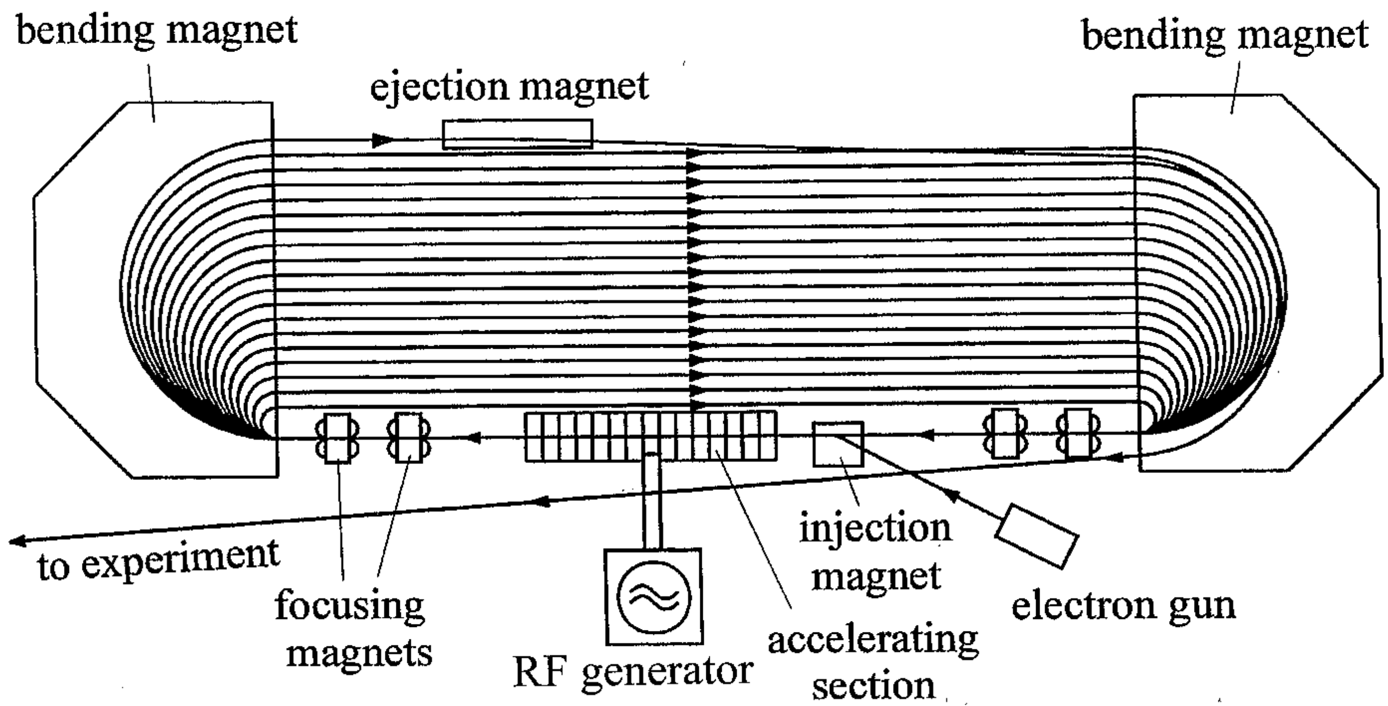
The microtron

- Electrons are quickly relativistic and cannot be accelerated in a cyclotron.
- In a microtron the revolution frequency changes, but each electron misses an integer number of RF waves.



- Today: Used for medical applications with one magnet and 20MeV.
- Nuclear physics: MAMI designed for 820MeV as race track microtron.

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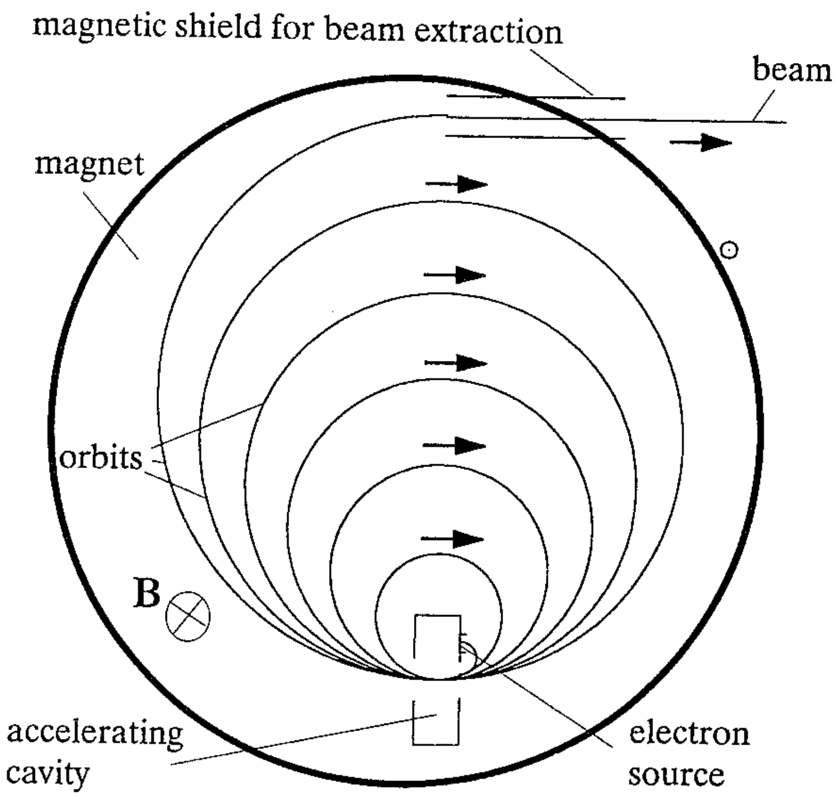
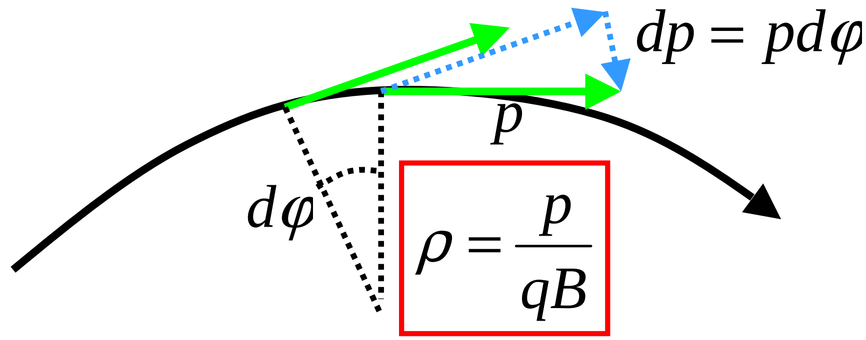
The microtron condition

- The extra time that each turn takes must be a multiple of the RF period.

$$\frac{dp}{dt} = qvB \Rightarrow \rho = \frac{dl}{d\varphi} = \frac{vdt}{dp/p} = \frac{p}{qB}$$





$$\Delta t = 2\pi\left(\frac{\rho_{n+1}}{v_{n+1}} - \frac{\rho_n}{v_n}\right)$$

$$= \frac{2\pi}{qB}(m_0\gamma_{n+1} - m_0\gamma_n) = \frac{2\pi}{qBc^2}\Delta K$$

$$\Delta K = n\frac{qBc^2}{\omega_{RF}} \quad \text{for an integer n}$$

B=1T, n=1, and f_{RF}=3GHz leads to 4.78MeV
This requires a small linear accelerator.

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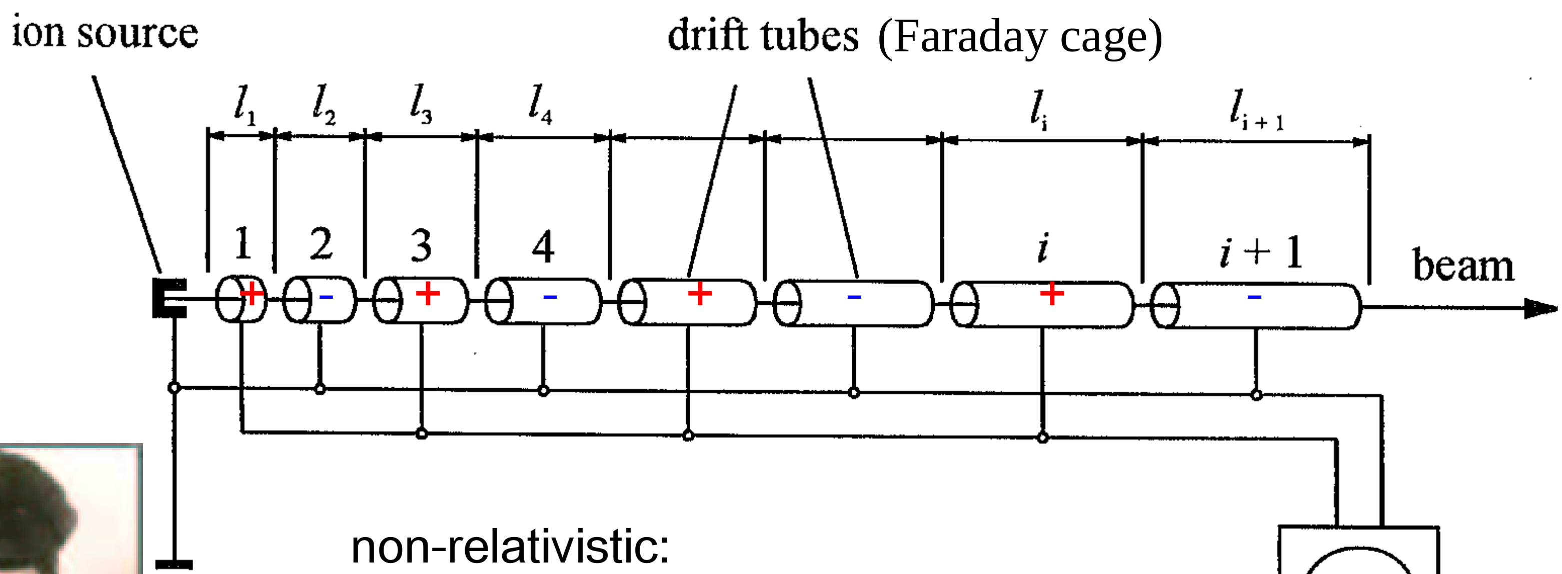
Wideroe linear accelerator

- 1924: Ising proposes a drift tube linear accelerator
- 1928: Wideroe builds the first drift tube linear accelerator for Na^+ and K^+




Wideroe

non-relativistic:

$$K_n = nqU_{\max}\sin\psi_0 = \tfrac{1}{2}mv_n^2$$

$$l_n = \tfrac{1}{2}v_n T_{RF} = \tfrac{1}{2}\beta_n\lambda_{RF} \propto \sqrt{n}$$

Called the π or the $1/2\beta\lambda$ mode

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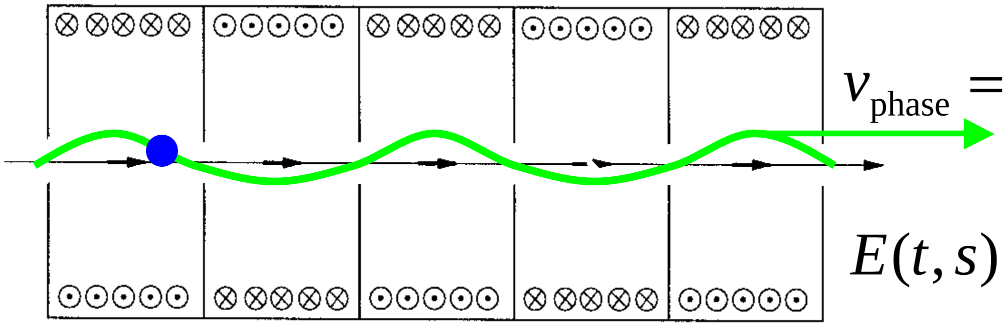
Accelerating cavities

- 1933: J.W. Beams uses resonant cavities for acceleration

Traveling wave cavity:

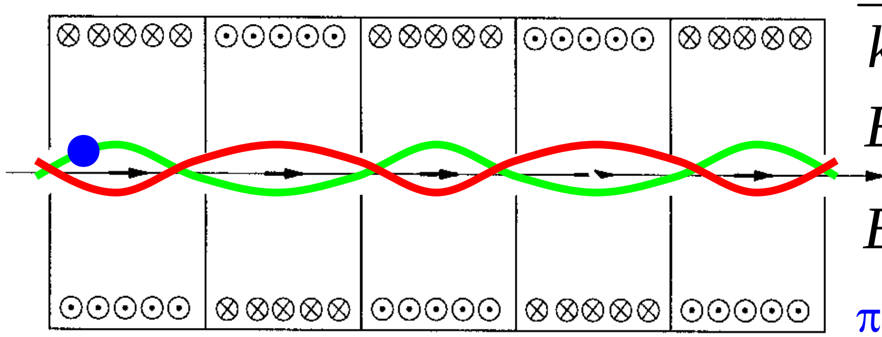


$$v_{\text{phase}} = \frac{\omega}{k} = v_{\text{particle}}$$

Here v=c for electrons

$$E(t,s) \approx E_{\max} \sin(\omega t - ks)$$

Standing wave cavity:



$$\frac{\omega}{k} = v_{\text{particle}}$$

$$E(t,s) \approx E_{\max} \sin(\omega t)\sin(ks)$$

$$E\left(\frac{s}{v_{\text{particle}}}, s\right) \approx E_{\max} \sin^2(ks)$$

π or the $1/2\beta\lambda$ mode

Transit factor (for this example): $\langle E \rangle = \frac{1}{\lambda_{RF}} \int_0^{\lambda_{RF}} E\left(\frac{s}{v_{\text{particle}}}, s\right) ds = \frac{1}{2} E_{\max}$

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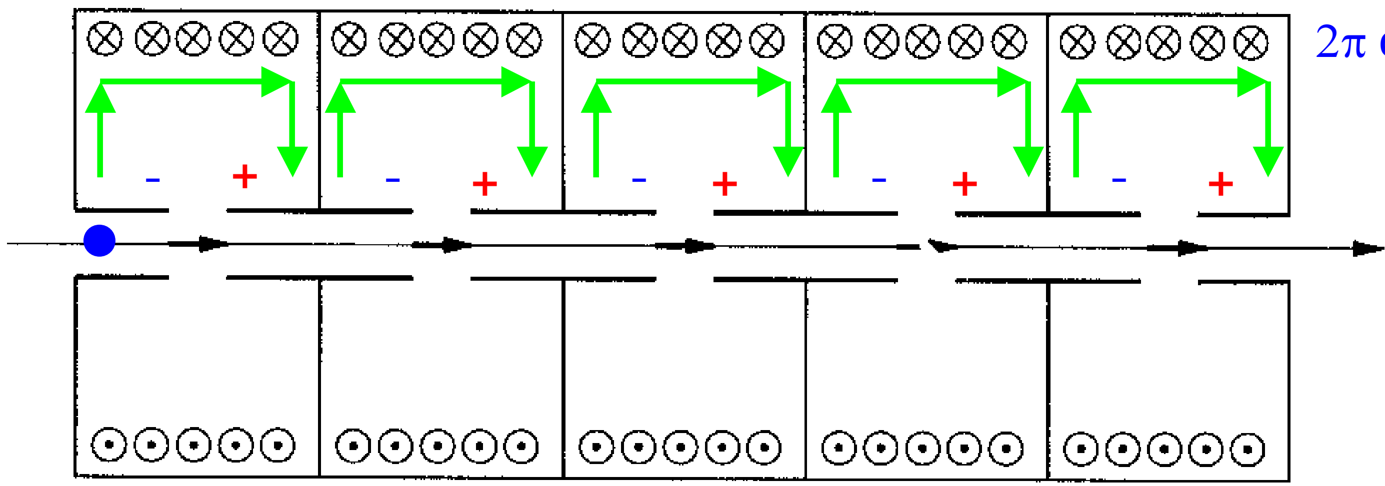
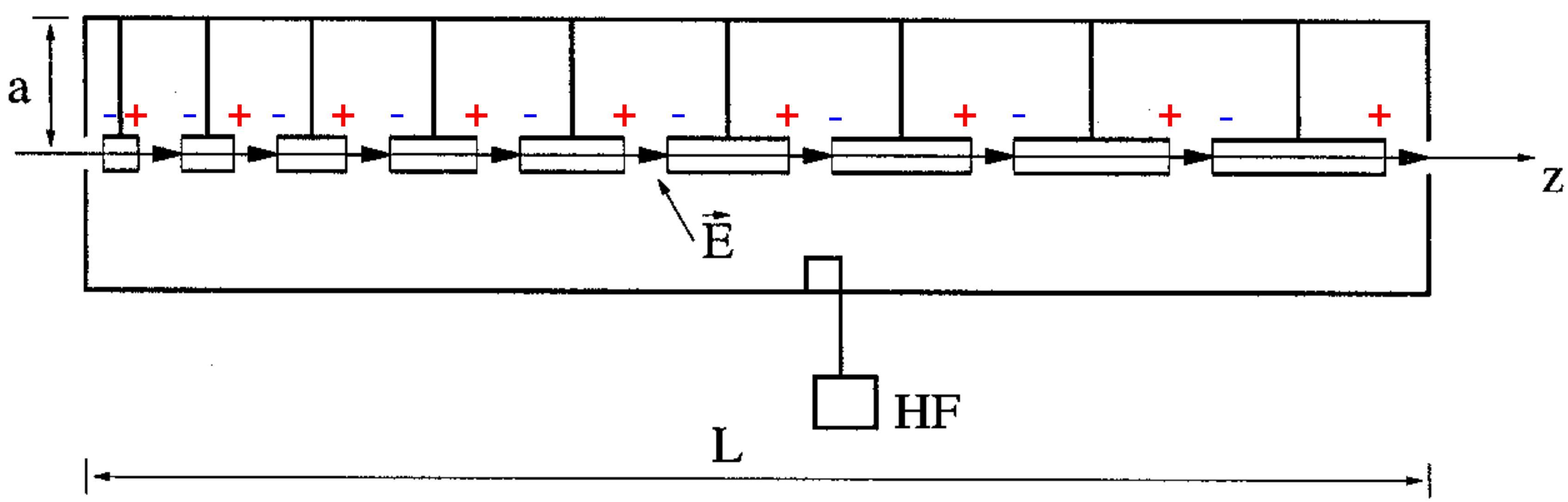
The Alvarez Linear Accelerator





Needs only one power input coupler and walls do not dissipate energy.

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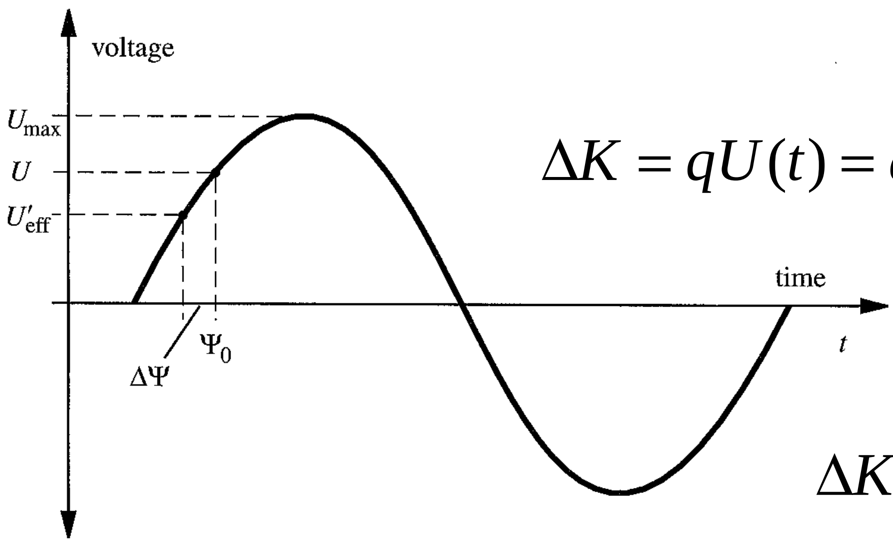
Phase focusing

- 1945: Veksler (UDSSR) and McMillan (USA) realize the importance of phase focusing



$$\Delta K = qU(t) = qU_{max}\sin(\omega(t-t_0)+\psi_0)$$

Longitudinal position in the bunch:

$$\sigma = s - s_0 = -v_0(t-t_0)$$

$$\Delta K(\sigma) = qU_{max}\sin(-\tfrac{\omega}{v_0}(s-s_0)+\psi_0)$$

$$\Delta K(0) > 0$$ (Acceleration)

$$\Delta K(\sigma) < \Delta K(0) \text{ for } \sigma > 0 \Rightarrow \tfrac{d}{d\sigma}\Delta K(\sigma) < 0$$ (Phase focusing)

$$\left.\begin{array}{r} qU(t) > 0 \\ q\tfrac{d}{dt}U(t) > 0 \end{array}\right\} \quad \psi_0 \in (0, \frac{\pi}{2})$$

Phase focusing is required in any RF accelerator.

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The RF quadrupole (RFQ)

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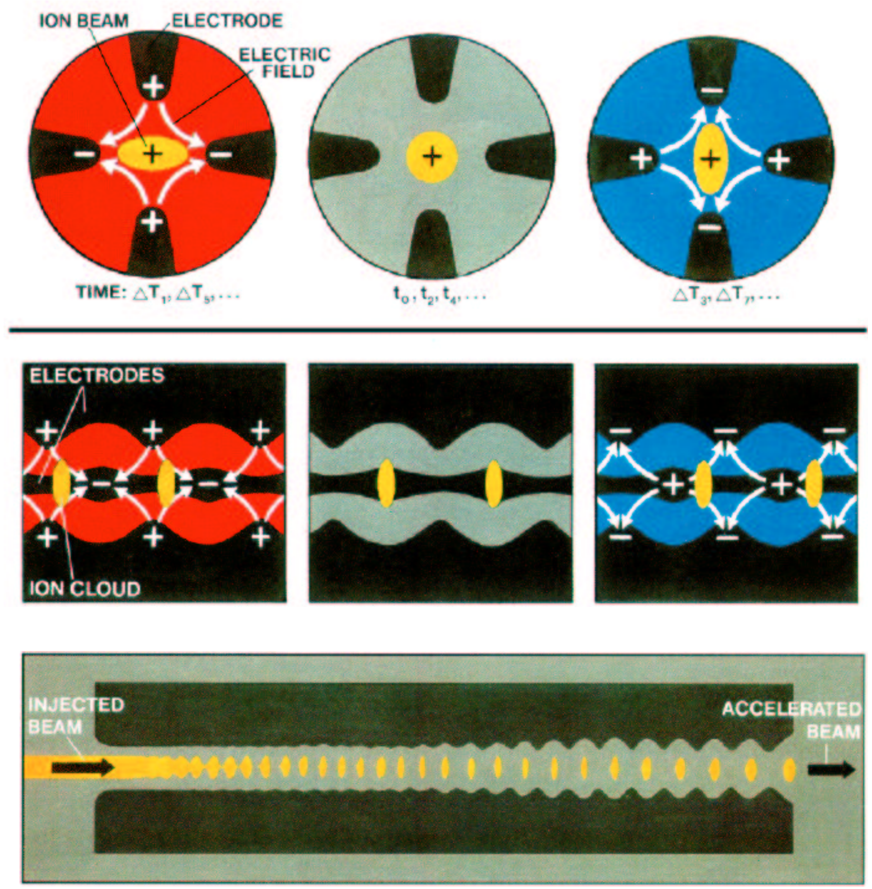
- 1970: Kapchinskii and Teplyakov invent the RFQ



Three historic lines of accelerators

Direct Voltage Accelerators Resonant Accelerators Transformer Accelerator

- 1924: Wideroe invents the betatron
- 1940: Kerst and Serber build a betatron for 2.3MeV electrons and understand betatron (transverse) focusing (in 1942: 20MeV)

Betatron:

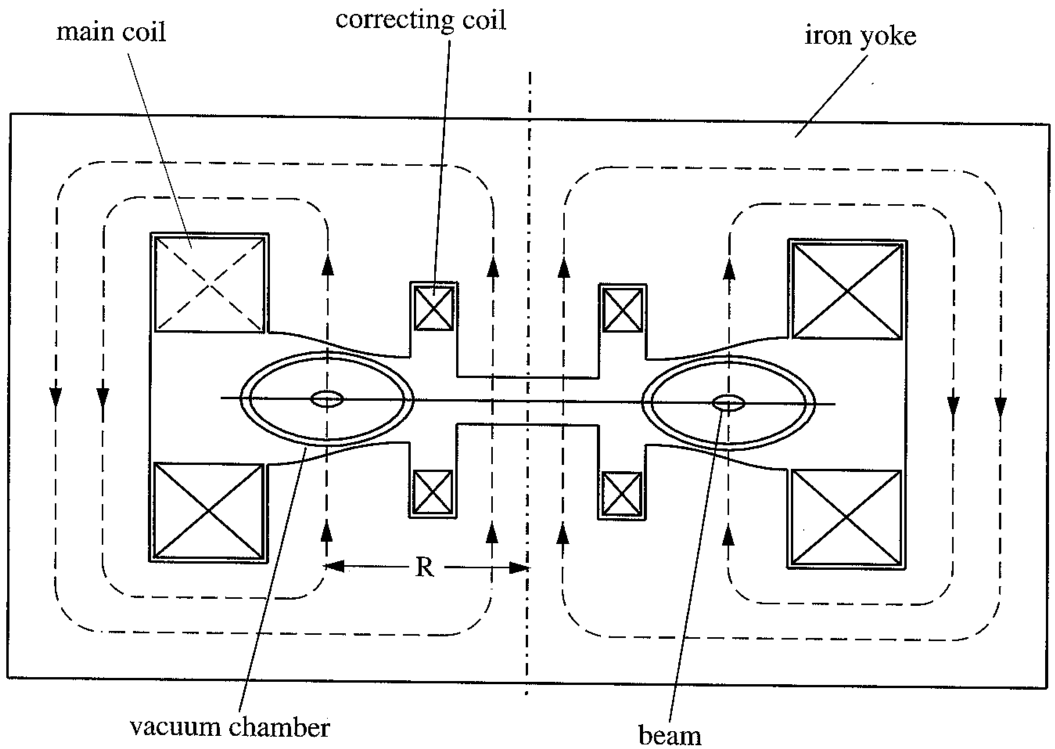
R=const, B=B(t)

Whereas for a cyclotron:

R(t), B=const

No acceleration section is needed since

$$\oint_{\partial A} \vec{E}\cdot d\vec{s} = -\iint_A \tfrac{d}{dt}\vec{B}\cdot d\vec{a}$$



The Betatron Condition

Condition: $R = \frac{-p_\varphi(t)}{qB_z(R,t)} = \text{const.}$ given $\oint_{\partial A} \vec{E} \cdot d\vec{s} = -\iint_A \frac{d}{dt} \vec{B} \cdot d\vec{a}$

$$E_\varphi(R,t) = -\frac{1}{2\pi R} \int \frac{d}{dt} B_z(r,t) r dr d\varphi = -\frac{R}{2} \left\langle \frac{d}{dt} B_z \right\rangle$$

$$\frac{d}{dt} p_\varphi(t) = qE_\varphi(R,t) = -q \frac{R}{2} \left\langle \frac{d}{dt} B_z \right\rangle$$

$$p_\varphi(t) = p_\varphi(0) - q \frac{R}{2} [\left\langle \frac{d}{dt} B_z \right\rangle(t) - \left\langle \frac{d}{dt} B_z \right\rangle(0)] = -RqB_z(R,t)$$

$$B_z(R,t) - B_z(R,0) = \frac{1}{2} [\left\langle \frac{d}{dt} B_z \right\rangle(t) - \left\langle \frac{d}{dt} B_z \right\rangle(0)]$$

Small deviations from this condition lead to transverse beam oscillations called betatron oscillations in all accelerators.

- Today: Betatrons with typically about 20MeV for medical applications

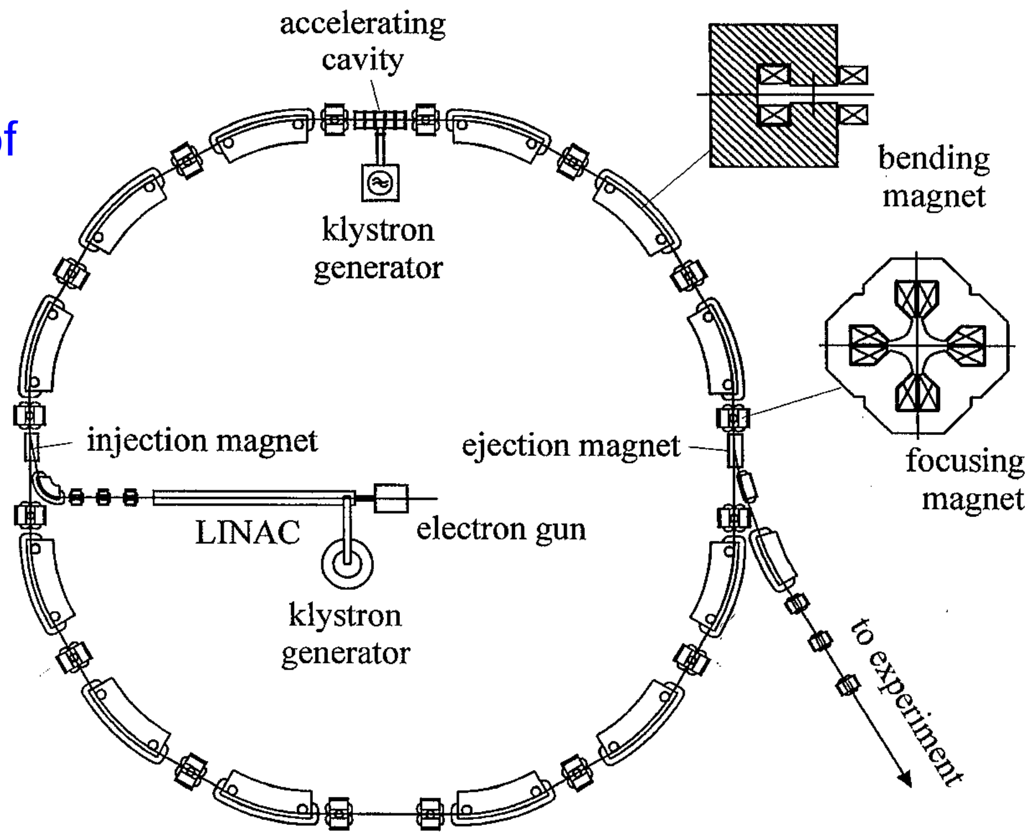
The Synchrotron

- 1945: Veksler (UDSSR) and McMillan (USA) invent the synchrotron
- 1946: Goward and Barnes build the first syncrotron (using a betatron magnet)
- 1949: Wilson et al. at Cornell are first to store beam in a synchtotron (later 300MeV, magnet of 80 Tons)
- 1949: McMillan builds a 320MeV electron synchrotron

- Many smaller magnets instead of one large magnet
- Only one acceleration section is needed, with

$$\omega = 2\pi \frac{v_{\text{particle}}}{L} n$$

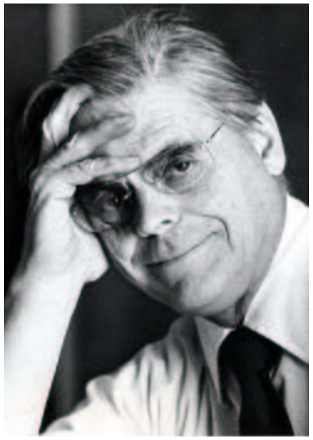
for an integer n called the harmonic number



Rober R Wilson, Architecture

Science Ed Center, FNAL (1990)

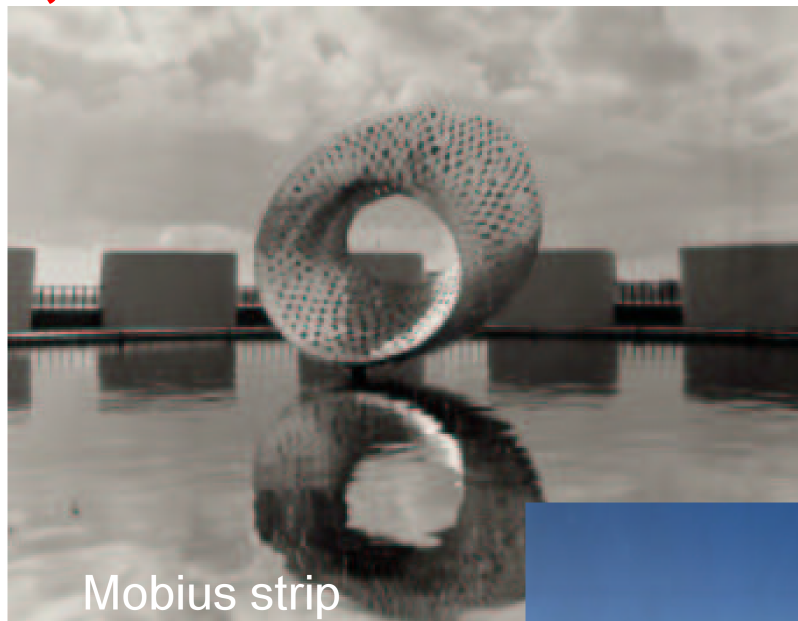
Wilson Hall, FNAL



Robert R Wilson
USA 1914-2000

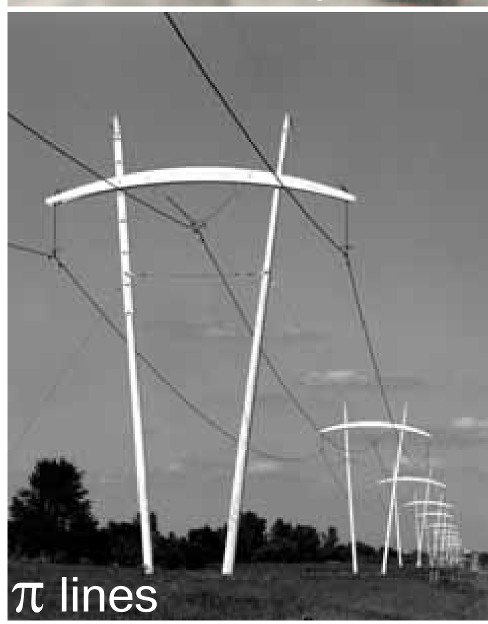
Rober R Wilson, Cornell & FNAL

Broken symmetry



Mobius strip

Tractricious



π lines

Hyperbolic obelisk

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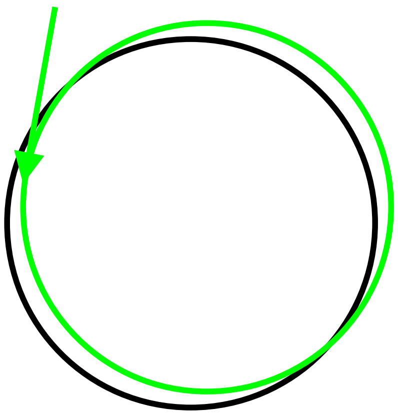
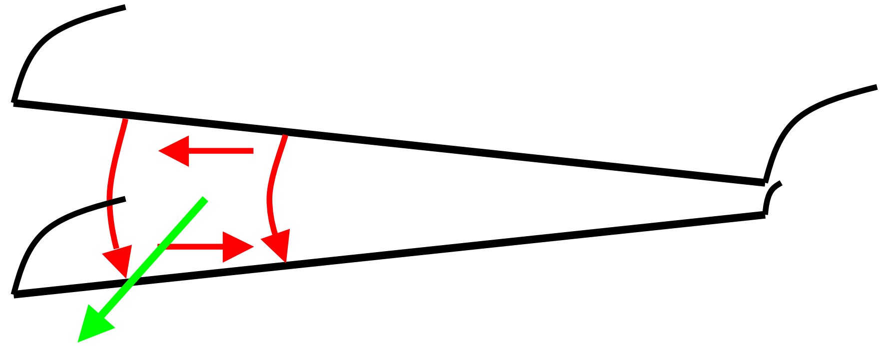
Weak focusing Synchrotrons

- 1952: Operation of the Cosmotron, 3.3 GeV proton synchrotron at Brookhaven
  Beam pipe height: 15cm.

Natural ring focusing:

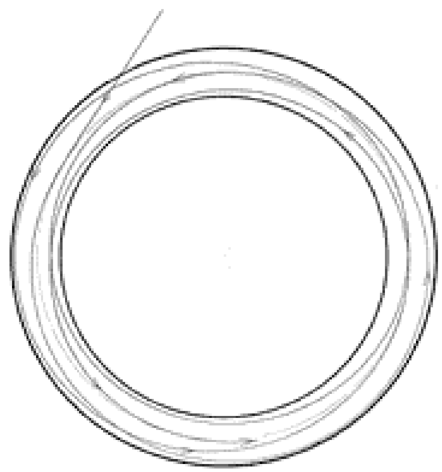




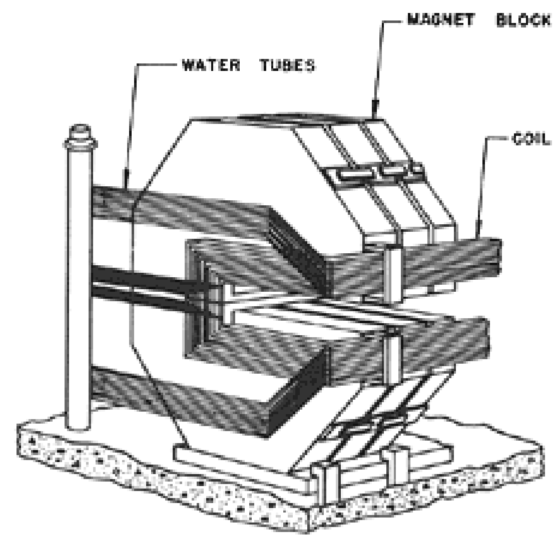
Vertical focusing
+ Horizontal defocusing + ring focusing

Focusing in both planes

The Cosmotron



Weak focusing accelerator



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Strong focusing Synchrotrons

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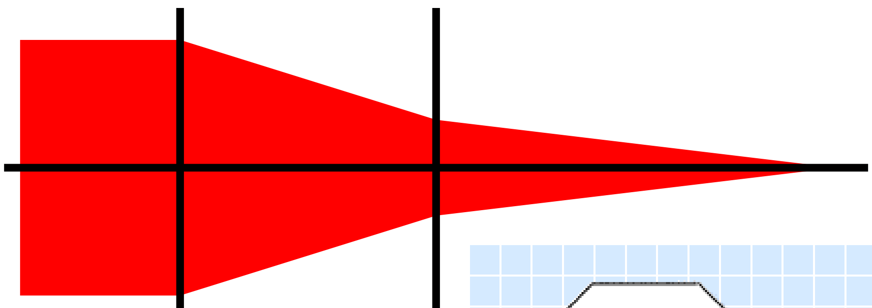
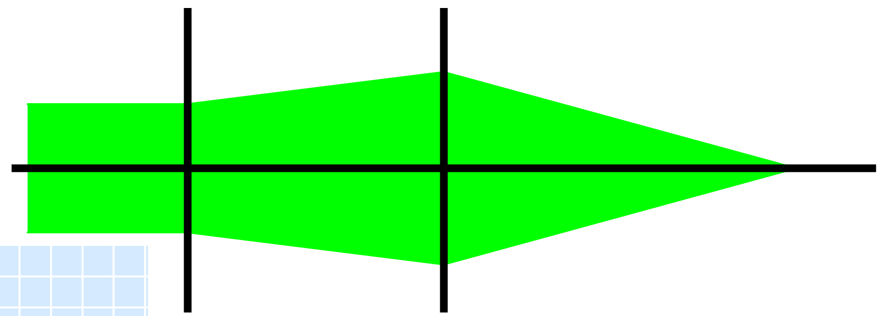
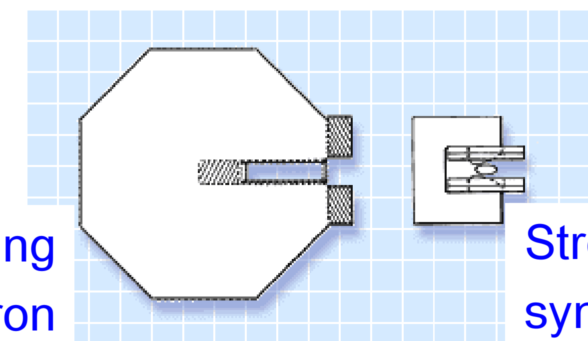
- 1952: Courant, Livingston, Snyder publish about strong focusing
- 1954: Wilson et al. build first synchrotron with strong focusing for 1.1MeV electrons at Cornell, 4cm beam pipe height, only 16 Tons of magnets.
- 1959: CERN builds the PS for 28GeV after proposing a 5GeV weak focusing accelerator for the same cost (still in use)

Transverse fields defocus in one plane if they focus in the other plane.
But two successive elements, one focusing the other defocusing,
can focus in both planes:







Weak focusing synchrotron

Strong focusing synchrotron

- Today: only strong focusing is used. Due to bad field quality at lower field excitations the injection energy is 20-500MeV from a linac or a microtron.

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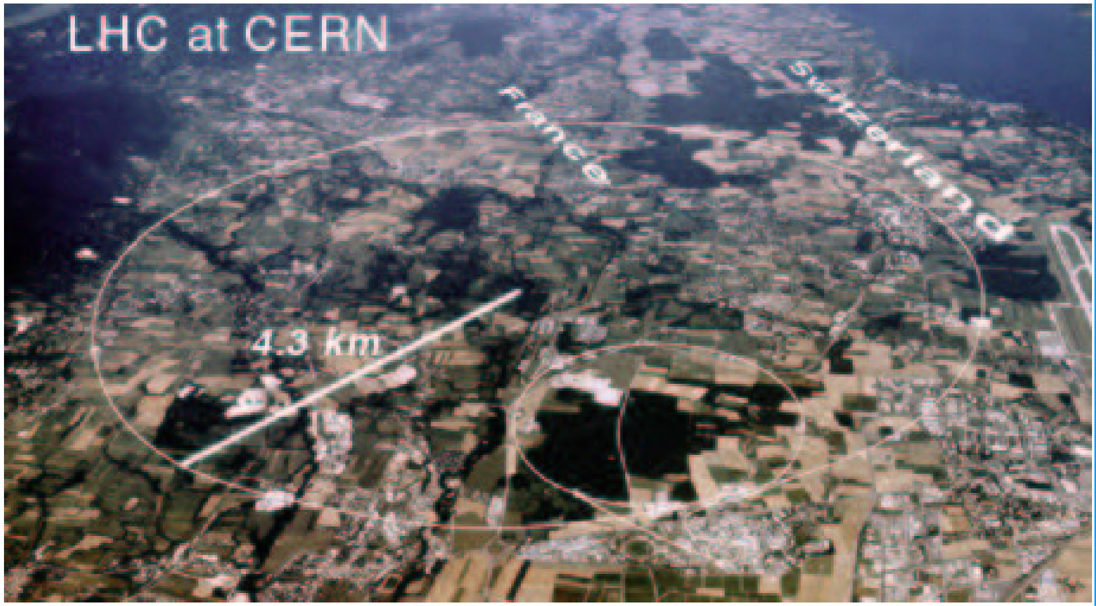
Limits of Synchrotrons

$$\rho = \frac{p}{qB} \quad \Rightarrow$$ The rings become too long

Protons with p = 20 TeV/c , B = 6.8 T would require a 87 km SSC tunnel
Protons with p = 7 TeV/c , B = 8.4 T require CERN's 27 km LHC tunnel

$$P_{\text{radiation}} = \frac{c}{6\pi\varepsilon_0} N \frac{q^2}{\rho^2} \gamma^4 \quad \Downarrow$$

Energy needed to compensate
Radiation becomes too large



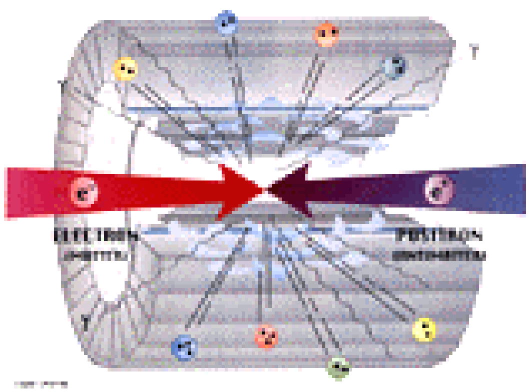
Electron beam with p = 0.1 TeV/c in CERN's 27 km LEP tunnel radiated 20 MW
Each electron lost about 4GeV per turn, requiring many of RF accelerating sections.

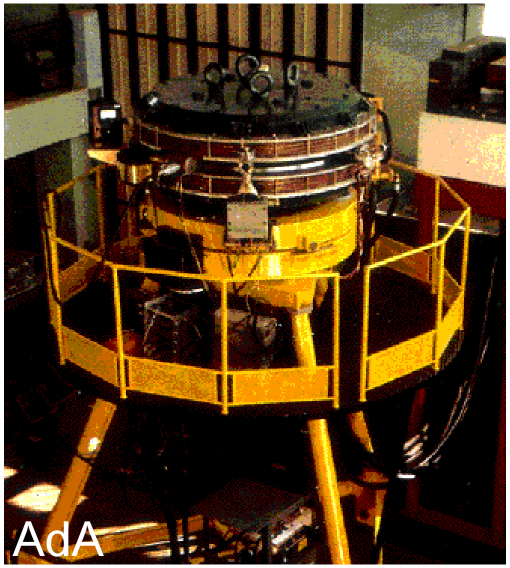
Colliding Beam Accelerators

- 1961: First storage ring for electrons and positrons (AdA) in Frascati for 250MeV
- 1972: SPEAR electron positron collider at 4GeV. Discovery of the J/Psi at 3.097GeV by Richter (SPEAR) and Ting (AGS) starts the November revolution and was essential for the quarkmodel and chromodynamics.
- 1979: 5GeV electron positron collider CESR (designed for 8GeV)

Advantage:

More center of mass energy





Drawback:

Less dense target

The beams therefore must be stored for a long time.

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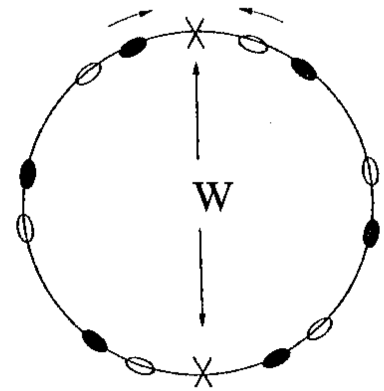
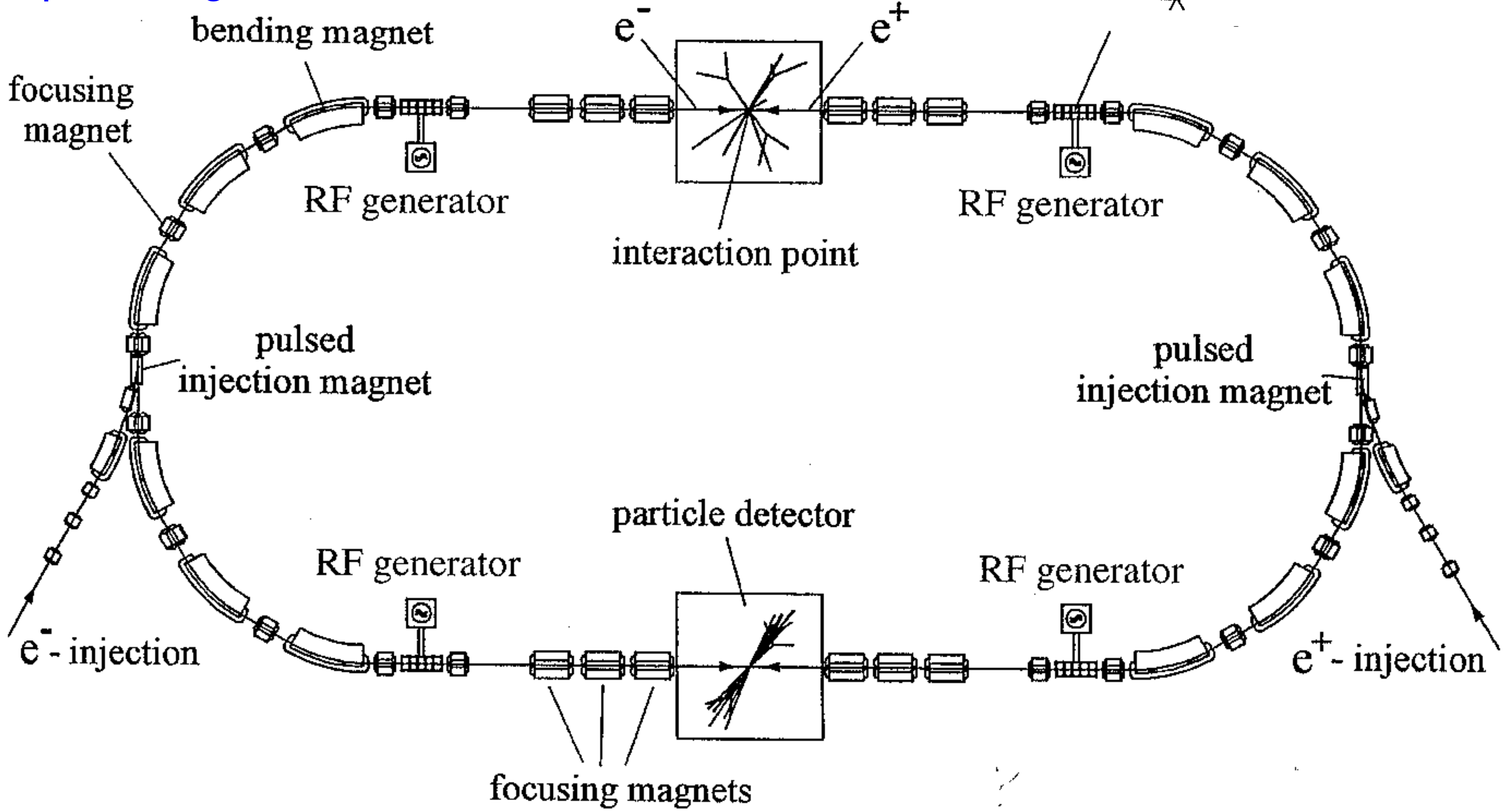
Ellements of a Collider



Chalenges:

- Saving one beam while injection another
- Avoiding collisions outside the detectors.
- Compensating the forces between e^+ and e^- beams



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Storage Rings

To avoid the loss of collision time during filling of a synchrotron, the beams in colliders must be stored for many milions of turns.

Chalenges:

- Required vacuum of pressure below 10^{-7} Pa = 10^{-9} mbar, 3 orders of magnitude below that of other accelerators.
- Fields must be stable for a long time, often for hours.
- Field errors must be small, since their effect can add up over millions of turns.
- Even though a storage ring does not accelerate, it needs acceleration sections for phase focusing and to compensate energy loss due to the emission of radiation.

Further Development of Colliders

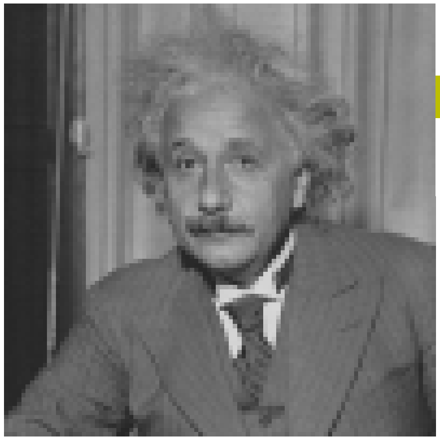
- 1981: Rubbia and van der Meer use stochastic cooling of antiportons and discover W+,W- and Z vector bosons of the weak interaction
- 1987: Start of the superconducting TEVATRON at FNAL
- 1989: Start of the 27km long LEP electron positron collider
- 1990: Start of the first asymmetric collider, electron (27.5GeV) proton (920GeV) in HERA at DESY
- 1998: Start of asymmetric two ring electron positron colliders KEK-B / PEP-II
- Today: 27km, 7 TeV proton collider LHC being build at CERN

NP 1984
Carlo Rubbia
Italy 1934 -

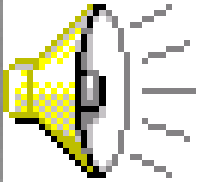
NP 1984
Simon van der Meer
Netherlands 1925 -

Special Relativity

$$E = mc^2$$



Albert Einstein, 1879-1955
Nobel Prize, 1921
Time Magazine Man of the Century



Four-Vectors:

Quantities that transform according to the Lorentz transformation when viewed from a different inertial frame.

Examples:

$$X^\mu \in \{ct, x, y, z\}$$

$$P^\mu \in \{\tfrac{1}{c}E, p_x, p_y, p_z\}$$

$$\Phi^\mu \in \{\tfrac{1}{c}\phi, A_x, A_y, A_z\}$$

$$J^\mu \in \{c\rho, j_x, j_y, j_z\}$$

$$K^\mu \in \{\tfrac{1}{c}\omega, k_x, k_y, k_z\}$$

$$X^\mu \in \{ct, x, y, z\} \quad \Rightarrow \quad X^\mu X_\mu = (ct)^2 - \vec{x}^{\,2} = \text{const.}$$

$$P^\mu \in \{\tfrac{1}{c}E, p_x, p_y, p_z\} \Rightarrow \quad P^\mu P_\mu = \left(\frac{E}{c}\right)^2 - \vec{p}^{\,2} = (m_0 c)^2 = \text{const.}$$

Available Energy

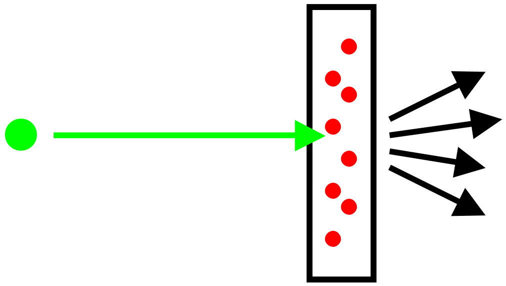
$$\tfrac{1}{c^2}E_{\text{cm}}^2 = (P_1^\mu + P_2^\mu)_{\text{cm}}(P_{1\mu} + P_{2\mu})_{\text{cm}}$$

$$= (P_1^\mu + P_2^\mu)(P_{1\mu} + P_{2\mu})$$

$$= \tfrac{1}{c^2}(E_1 + E_2)^2 - (p_{z1} - p_{z2})^2$$

$$= 2(\frac{E_1 E_2}{c^2} + p_{z1}p_{z2}) + (m_{01}c)^2 + (m_{02}c)^2$$

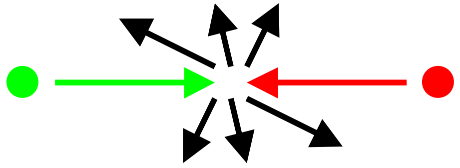
Operation of synchrotrons: fixed target experiments where some energy is in the motion of the center off mass of the scattering products



$$E_1 >> m_{01}c^2, m_{02}c^2; p_{z2} = 0; E_2 = m_{02}c^2 \quad \Rightarrow \quad E_{\text{cm}} = \sqrt{2E_1 m_{02}c^2}$$

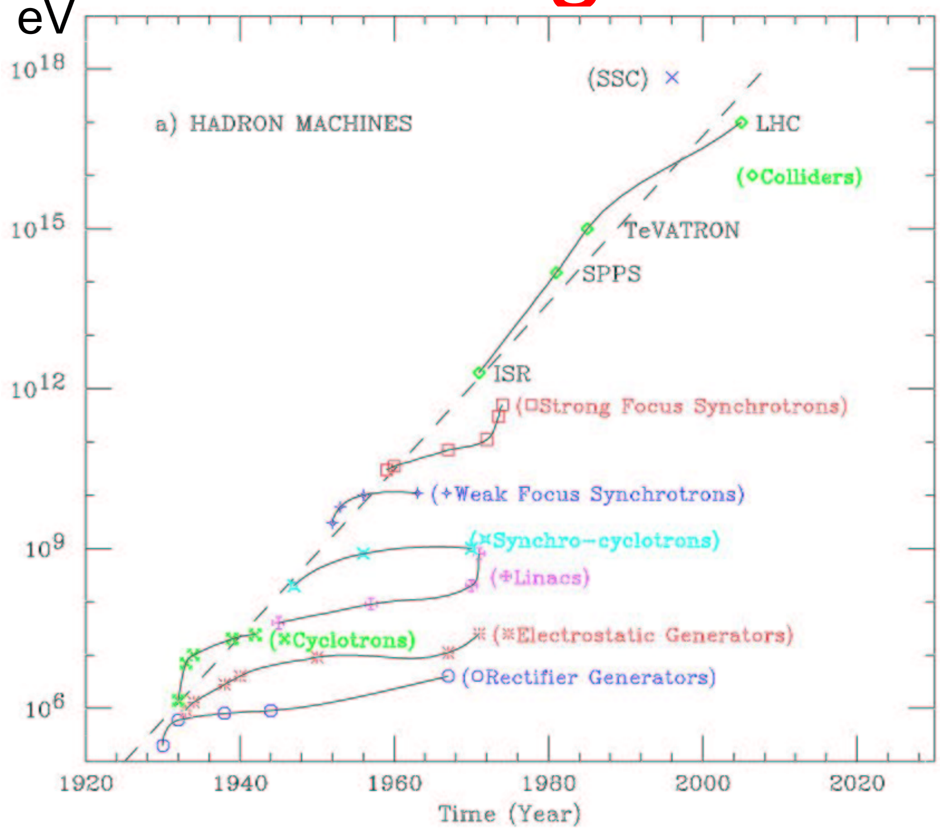
Operation of colliders:

the detector is in the center of mass system



$$E_1 >> m_{01}c^2; E_2 >> m_{02}c^2 \quad \Rightarrow \quad E_{\text{cm}} = 2\sqrt{E_1 E_2}$$

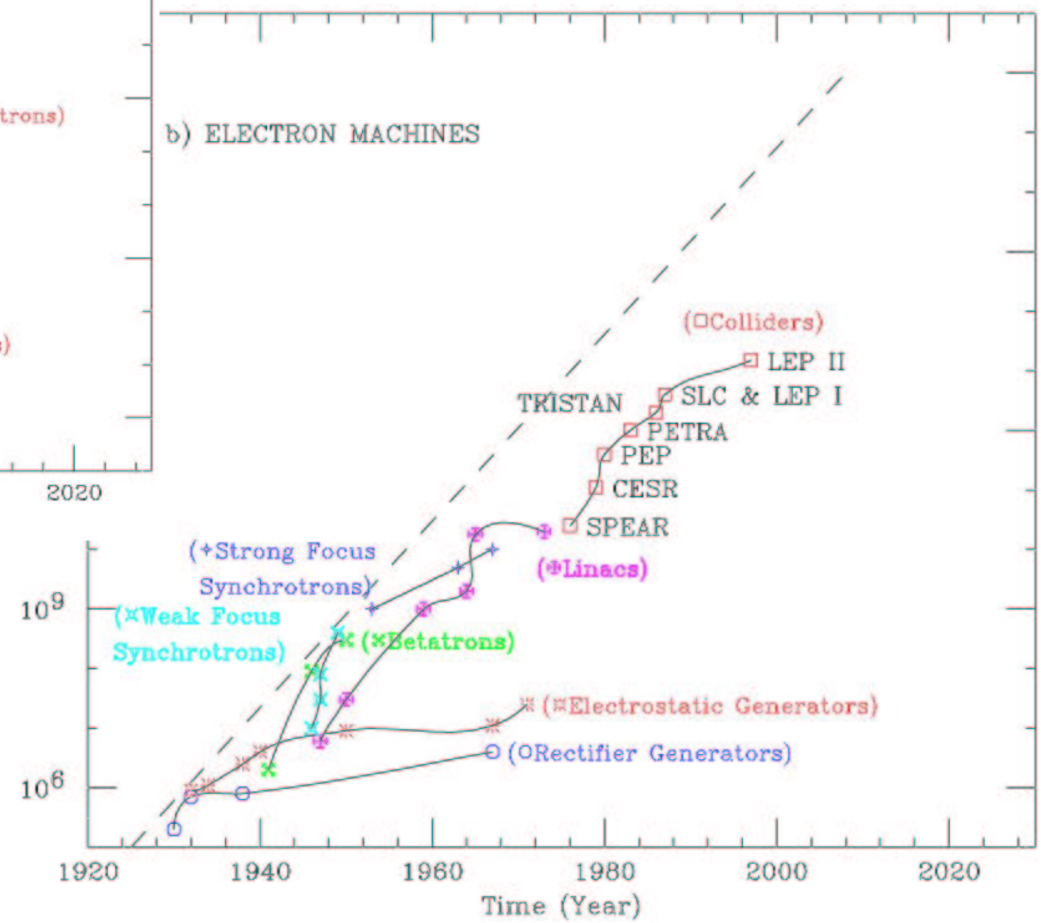
The Livingston Chart



Comparison:

highest energy cosmic rays

have a few 10^{20}eV



Energy that would be needed in a fixed target experiment versus the year of achievement

$$E_1 = \frac{E_{\text{cm}}^2}{2 m_{02} c^2}$$

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Example: Production of the pbar

- 1954: Operation of Bevatron, first proton synchrotron for 6.2GeV, production of the antiporton by Chamberlain and Segrè

$$p + p \quad \mapsto \quad p + p + p + \overline{p}$$

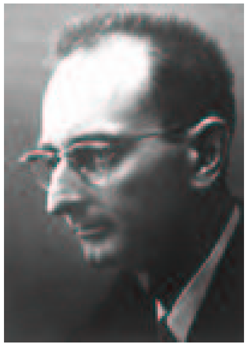
$$\tfrac{1}{c^2} E_{\text{cm}}^2 = 2(\frac{E_1 E_2}{c^2} + p_{z1} p_{z2}) + (m_{01} c)^2 + (m_{02} c)^2$$

$$(4m_{p0}c)^2 < \tfrac{1}{c^2} E_{\text{cm}}^2 = 2\frac{E_1 m_{p0}}{c^2} + (m_{p0}c)^2 + (m_{p0}c)^2$$

$$7 m_{p0} c^2 < E_1$$

$$\underline{K_1} = E_1 - m_0 c^2 > 6 m_{p0} c^2 \underline{= 5.628\,\text{GeV}}$$

NP 1959
Emilio Gino Segrè
Italy 1905 – USA 1989



NP 1959
Owen Chamberlain
USA 1920 -

Example: c-cbar states

- 1974: Observation of $c-\bar{c}$ resonances (J/Ψ) at Ecm = 3095MeV at the e^+/e^- collider SPEAR

$$\frac{1}{c^2}E_{\text{cm}}^2 = 2(\frac{E_1E_2}{c^2} + p_{z1}p_{z2}) + (m_{01}c)^2 + (m_{02}c)^2$$

$$E_1 = E_2 \quad \Rightarrow \quad E_{\text{cm}}^2 = 4E^2$$

Energy per beam: $K = E - m_0 c = 1547\text{MeV}$

Beam energy needed for an equivalent fixed target experiment: $\frac{E_{cm}^2}{c^2} = 2[Em + (mc)^2]$

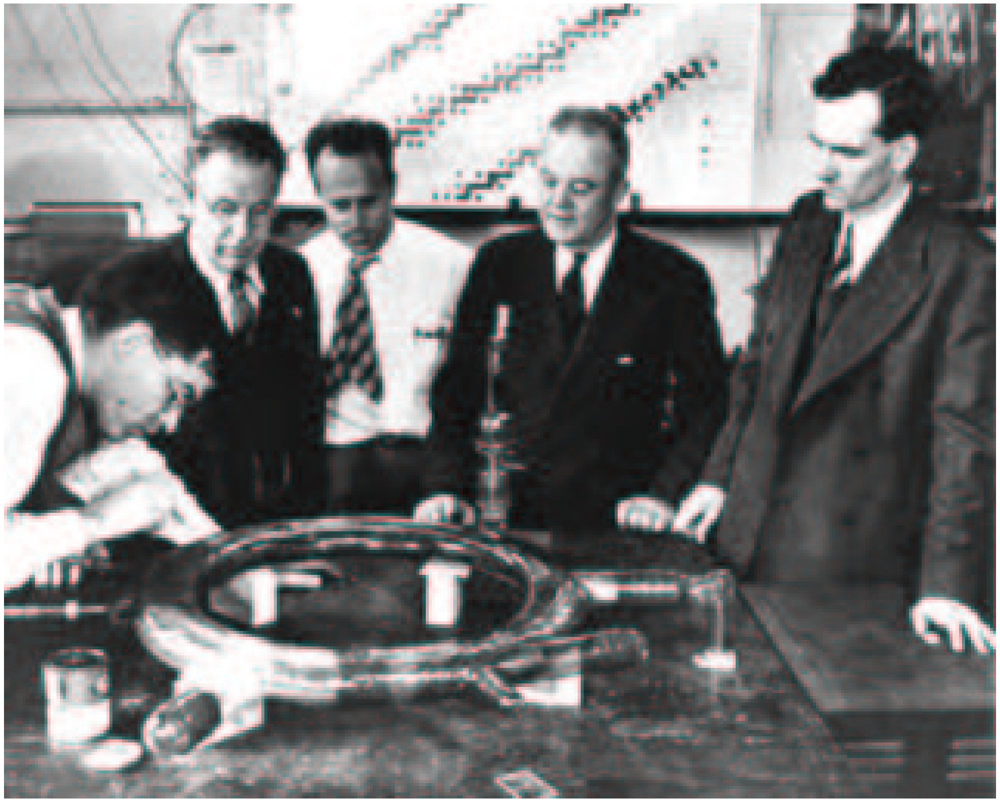
$$K = E - m_{0e}c^2 = \frac{E_{cm}^2 - 4(m_{0e}c^2)^2}{2m_{0e}c^2} = 9.4\text{TeV}$$

NP 1976
Burton Richter
USA 1931 -

NP 1976
Samuel CC Ting
USA 1936 -

Rings for Synchrotron Radiation

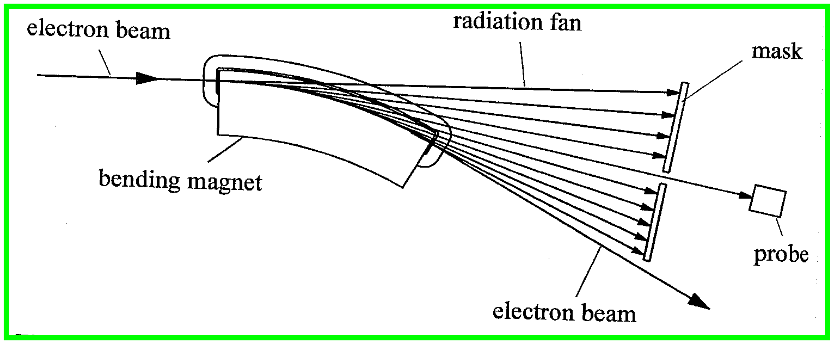
- 1947: First detection of synchrotron light at General Electrics.
- 1952: First accurate measurement of synchrotron radiation power by Dale Corson with the Cornell 300MeV synchrotron.
- 1968: TANTALOS, first dedicated storage ring for synchrotron radiation

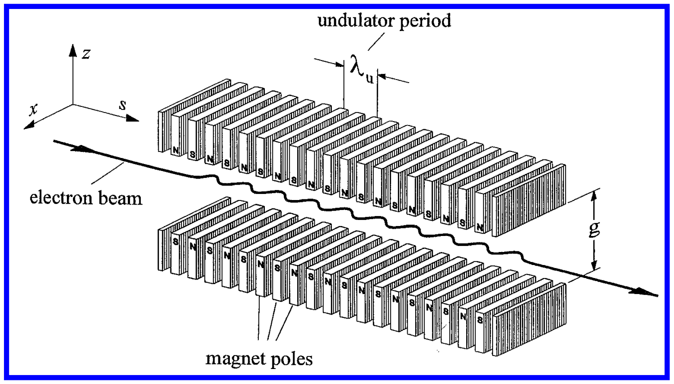


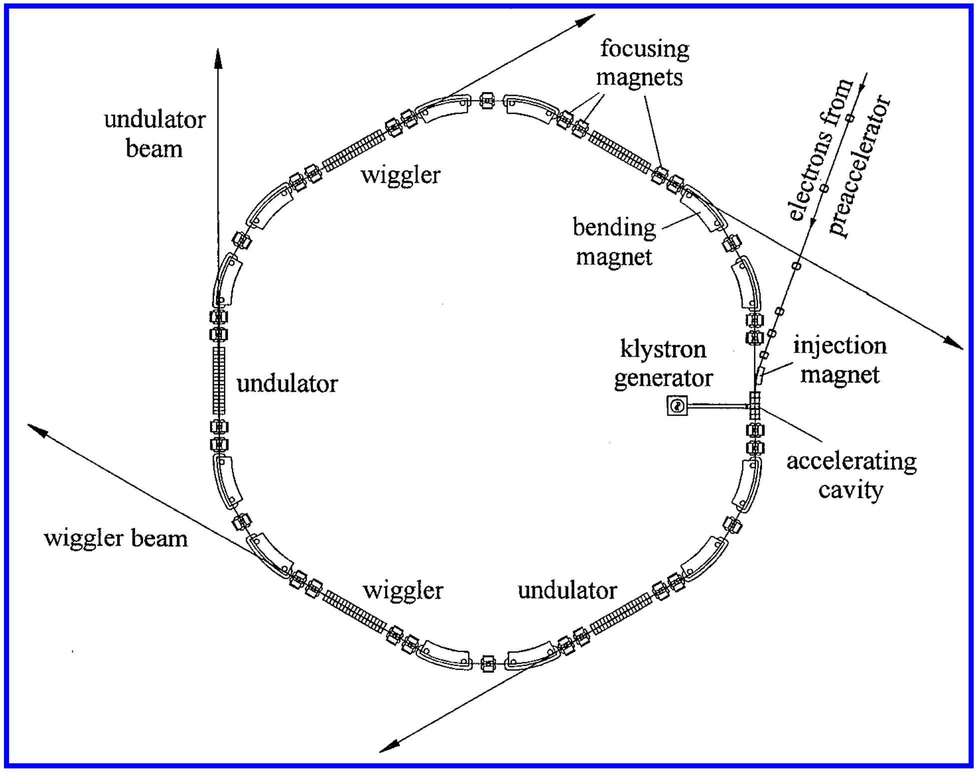
Dale Corson
Cornell's 8^{th} president
USA 1914 –

3 Generations of Light Sources

- 1st Genergation (1970s): Many HEP rings are parasitically used for X-ray production
- 2nd Generation (1980s): Many dedicated X-ray sources (light sources)
- 3rd Generation (1990s): Several rings with dedicated radiation devices (wigglers and undulators)
- Today (4th Generation): Construction of Free Electron Lasers (FELs) driven by LINACs







Accelerators of the World

09/04/03
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Sorted by Location

Europe

AGOR	Accelerateur Groningen-ORsay, KVI Groningen, Netherlands
ANKA	Ångströmquelle Karlsruhe, Karlsruhe, Germany (Forschungsgruppe Synchrotronstrahlung (FGS))
ASTRID	Aarhus Storage Ring in Denmark, ISA, Aarhus, Denmark
BESSY	Berliner Elektronenspeicherring-Gesellschaft für Synchrotronstrahlung, Germany (BESSY I status, BESSY II status)
BINP	Budker Institute for Nuclear Physics, Novosibirsk, Russian Federation (VEPP-2M collider, VEPP-4M collider (status))
CERN	Centre Europeen de Recherche Nucleaire, Geneva, Suisse (LEP & SPS Status, LHC, CLIC, PS-Division, SL-Division)
COSY	Cooler Synchrotron, IKP, FZ Jülich, Germany (COSY Status)
CYCLONE	Cyclotron of Louvain la Neuve, Louvain-la-Neuve, Belgium
DELTA	Dortmund Electron Test Accelerator, U of Dortmund, Germany (DELTA Status)
DESY	Deutsches Elektronen Synchrotron, Hamburg, Germany (HERA, PETRA and DORIS status, TESLA)
ELBE	ELectron source with high Brilliance and low Emittance, FZ Rossendorf, Germany
ELETTRA	Trieste, Italy (ELETTRA status)
ELSA	Electron Stretcher Accelerator, Bonn University, Germany (ELSA status)
ESRF	European Synchrotron Radiation Facility, Grenoble, France (ESRF status)
GANIL	Grand Accélérateur National d'Ions Lourds, Caen, France
GSI	Gesellschaft für Schwerionenforschung, Darmstadt, Germany
IHEP	Institute for High Energy Physics, Protvino, Moscow region, Russian Federation
INFN	Istituto Nazionale di Fisica Nucleare, Italy, LNF - Laboratori Nazionali di Frascati (DAFNE, other accelerators), LNL - Laboratori Nazionali di Legnaro (Tandem, CN Van de Graaff, AN 2000 Van de Graaff), LNS - Laboratori Nazionali del Sud, Catania, (Superconducting Collider & Van de Graaff Tandem)
ISIS	Rutherford Appleton Laboratory, Oxford, U.K. (ISIS Status)
ISL	IonenStrahlLabor am HMI, Berlin, Germany
JINR	Joint Institute for Nuclear Research, Dubna, Russian Federation (U-200, U-400, U-400M, Storage Ring, LHE Synchrophasotron / Nuclotron)
JYFL	Jyväskylän Yliopiston Fysiikan Laitos, Jyväskylä, Finland
KTH	Kungl Tekniska Högskola (Royal Institute of Technology), Stockholm, Sweden (Alfén Lab electron accelerators)

Accelerators of the World

09/04/03
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LMU/TUM	Accelerator of LMU and TU Muenchen, Munich, Germany
LURE	Laboratoire pour l'Utilisation du Rayonnement Electromagnétique, Orsay, France (DCI, Super-ACO status, CLIO)
MAMI	Mainzer Microtron, Mainz U, Germany
MAX-Lab	Lund University, Sweden
MSL	Manne Siegbahn Laboratory, Stockholm, Sweden (CRYRING)
NIKHEF	Nationaal Instituut voor Kernfysica en Hoge-Energie Fysica, Amsterdam, Netherlands (AmPS *closed*)
PSI	Paul Scherrer Institut, Villigen, Switzerland (PSI status, SLS *under construction*)
S-DALINAC	Darmstadt University of Technology, Germany (S-DALINAC status)
SRS	Synchrotron Radiation Source, Daresbury Laboratory, Daresbury, U.K. (SRS Status)
TSL	The Svedberg Laboratory, Uppsala University, Sweden (CELSIUS)
TSR	Heavy-Ion Test Storage Ring, Heidelberg, Germany

North America

88" Cycl.	88-Inch Cyclotron, Lawrence Berkeley Laboratory (LBL), Berkeley, CA
ALS	Advanced Light Source, Lawrence Berkeley Laboratory (LBL), Berkeley, CA (ALS Status)
ANL	Argonne National Laboratory, Chicago, IL (Advanced Photon Source APS [status], Intense Pulsed Neutron Source IPNS [status], Argonne Tandem Linac Accelerator System ATLAS)
BNL	Brookhaven National Laboratory, Upton, NY (AGS, ATF, NSLS, RHIC)
CAMD	Center for Advanced Microstructures and Devices
CHESS	Cornell High Energy Synchrotron Source, Cornell University, Ithaca, NY
CLS	Canadian Light Source, U of Saskatchewan, Saskatoon, Canada
CESR	Cornell Electron-positron Storage Ring, Cornell University, Ithaca, NY (CESR Status)
FNAL	Fermi National Accelerator Laboratory , Batavia, IL (Tevatron)
IAC	Idaho accelerator center, Pocatello, Idaho
IUCF	Indiana University Cyclotron Facility, Bloomington, Indiana
JLab	aka TJNAF, Thomas Jefferson National Accelerator Facility (formerly known as CEBAF), Newport News, VA
LAC	Louisiana Accelerator Center, U of Louisiana at Lafayette, Louisiana
LANL	Los Alamos National Laboratory
MIT-Bates	Bates Linear Accelerator Center, Massachusetts Institute of Technology (MIT)
NSCL	National Superconducting Cyclotron Laboratory, Michigan State University
ORNL	Oak Ridge National Laboratory (EN Tandem Accelerator), Oak Ridge, Tennessee
SBSL	Stony Brook Superconducting Linac, State University of New York (SUNY)
SLAC	Stanford Linear Accelerator Center (Linac, NLC - Next Linear Collider, PEP - Positron Electron Project (finished), PEP-II - asymmetric B Factory (in commissioning), SLC - SLAC Linear electron positron Collider, SPEAR - Stanford Positron Electron Asymmetric Ring (actually SPEAR-II, see SSRL), SSRL - Stanford Synchrotron Radiation Laboratory)
SNS	Spallation Neutron Source, Oak Ridge, Tennessee
SRC	Synchrotron Radiation Center, U of Wisconsin - Madison (Aladdin Status)
SURF II	Synchrotron Ultraviolet Radiation Facility, National Institute of Standards and Technology (NIST), Gaithersburg, Maryland
TASCC	Tandem Accelerator Superconducting Cyclotron (Canada) *(closed)*
TRIUMF	TRI-University Meson Facility / National Meson Research Facility, Vancouver, BC (Canada)

South America

LNLS	Laboratorio Nacional de Luz Sincrotron, Campinas SP, Brazil
TANDAR	Tandem Accelerator, Buenos Aires, Argentina

Asia

BEPC	Beijing Electron-Positron Collider, Beijing, China
KEK	National Laboratory for High Energy Physics ("Koh-Ene-Ken"), Tsukuba, Japan (KEK-B, PF, JLC)
NSC	Nuclear Science Centre, New Delhi, India (15 UD Pelletron Accelerator)
PLS	Pohang Light Source, Pohang, Korea
RIKEN	Institute of Physical and Chemical Research ("Rikagaku Kenkyusho"), Hirosawa, Wako, Japan
SESAME	Synchrotron-light for Experimental Science and Applications in the Middle East, Jordan (under construction)
SPring-8	Super Photon ring - 8 GeV, Japan
SRRC	Synchrotron Radiation Research Center, Hsinchu, Taiwan (SRRC Status)
UVSOR	Ultraviolet Synchrotron Orbital Radiation Facility, Japan
VECC	Variable Energy Cyclotron, Calcutta, India

Africa

NAC	National Accelerator Centre, Cape Town, South Africa

Sorted by Accelerator Type

Electrons

Stretcher Ring/Continuous Beam facilities

ELSA (Bonn U), JLab, MAMI (Mainz U), MAX-Lab, MIT-Bates, PSR (SAL), S-DALINAC (TH Darmstadt), SLAC

Accelerators of the World

Synchrotron Light Sources

ANKA (FZK), ALS (LBL), APS (ANL), ASTRID (ISA), BESSY, CAMD (LSU), CHESS (Cornell Wilson Lab), CLS (U of Saskatchewan), DELTA (U of Dortmund), ELBE (FZ Rossendorf), Elettra, ELSA (Bonn U), ESRF, HASYLAB (DESY), LURE, MAX-Lab, LNLS, NSLS (BNL), PF (KEK), UVSOR (IMS), PLS, S-DALINAC (TH Darmstadt), SESAME, SLS (PSI), SPEAR (SSRL, SLAC), SPring-8, SRC (U of Wisconsin), SRRC, SRS (Daresbury), SURF II (NIST)

Other

Alfén Lab (KTH), IAC

Protons

88" Cyclotron (LBL), CELSIUS (TSL), COSY (FZ Jülich), IPNS (ANL), ISL (HMI), ISIS, IUCF, LHC (CERN), NAC, PS (CERN), PSI, SPS (CERN)

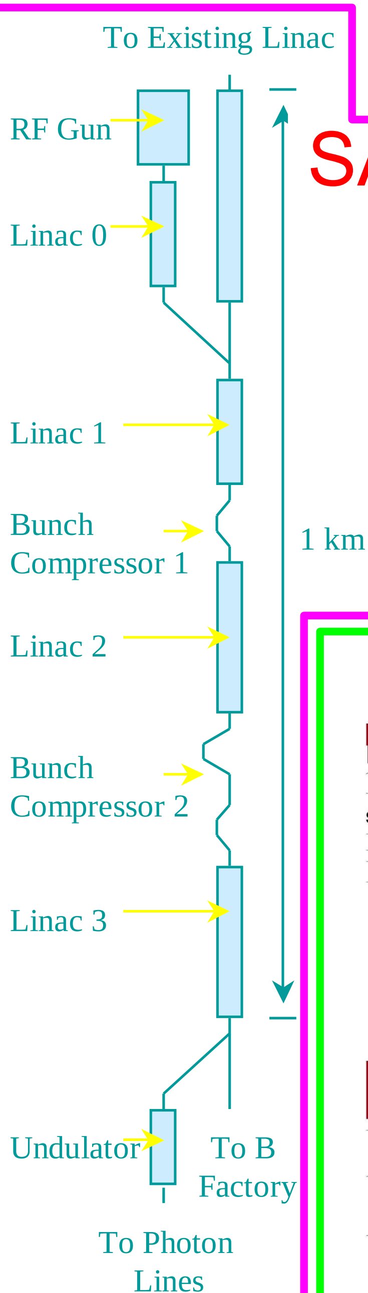
Light and Heavy Ions

88" Cyclotron (LBL), AGOR, ASTRID (ISA), ATLAS (ANL), CELSIUS (TSL), CRYRING (MSL), CYCLONE, EN Tandem (ORNL), GANIL, GSI, ISL (HMI), IUCF, JYFL, LAC, LHC (CERN), LHE Synchrophasotron / Nuclotron (JINR), LMU/TUM, LNL (INFN), LNS (INFN), NAC, NSC, PSI, RHIC (BNL), SBSL, SNS, SPS (CERN), TANDAR, TSR, U-200 / U-400 / U-400M / Storage Ring (JINR), VECC

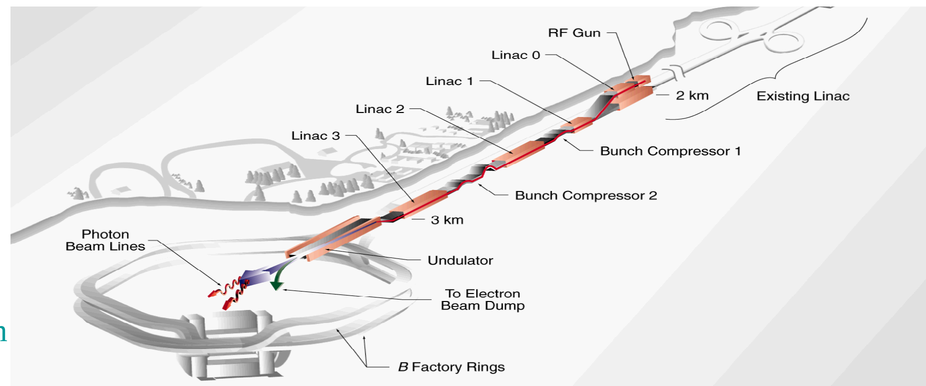
Collider

BEPC, CESR, DAFNE (LNF), HERA (DESY), LEP (CERN), LHC (CERN), PEP / PEP-II (SLAC), SLC (SLAC), KEK-B (KEK), TESLA (DESY), Tevatron (FNAL), VEPP-2M, VEPP-4M (BINP)

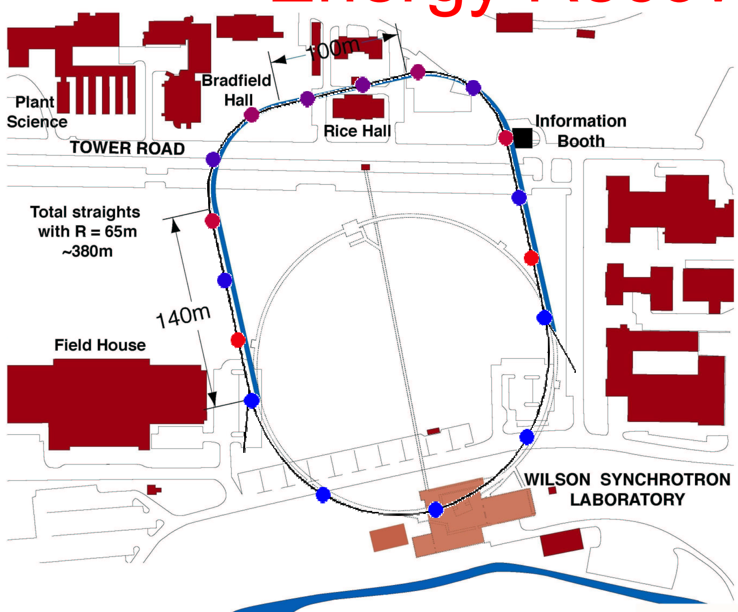
The Future



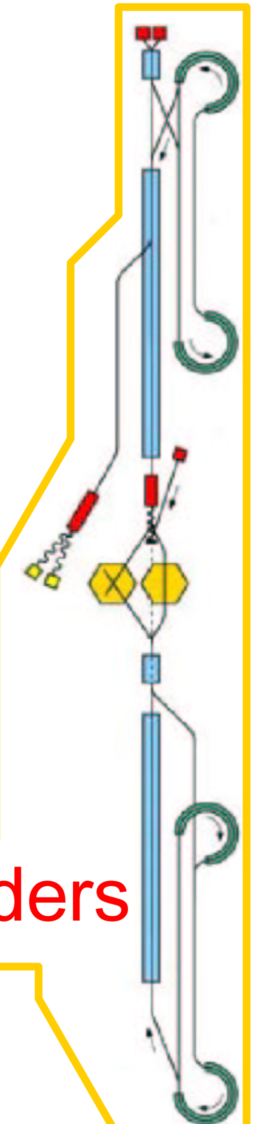
SASE Free Electron Lasers



Energy Recovery Linacs



Linear Colliders



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Macroscopic Fields in Accelerators

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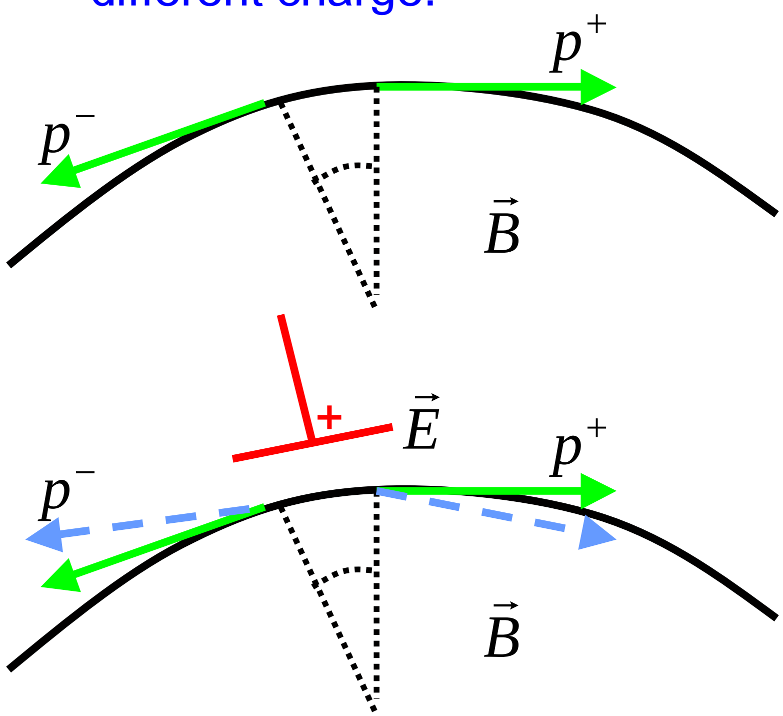
$$\frac{d}{dt}\vec{p} = q(\vec{E} + \vec{v} \times \vec{B})$$

E has a similar effect as v B.
For relativistic particles B = 1T has a similar effect as
E = cB = 3 10^8 V/m , such an
Electric field is beyond technical limits.

- Electric fields are only used for very low energies or
- For separating two counter rotating beams with different charge.



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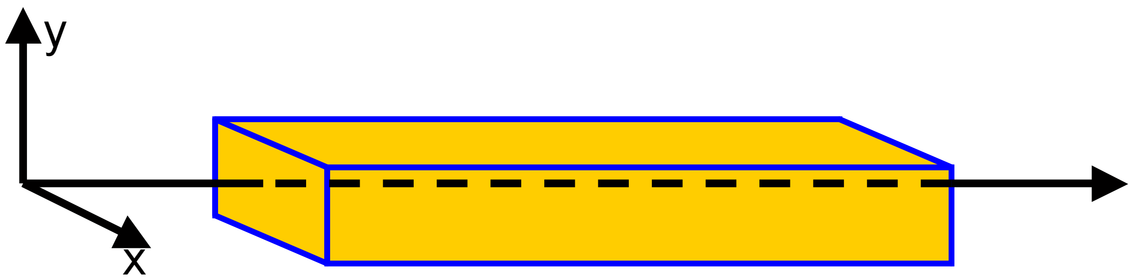
Magnetic Fields in Accelerators

Static magnetic fileds: $\partial_t \vec{B} = 0\,;\quad \vec{E} = 0$ Charge free space: $\vec{j} = 0$

$$\vec{\nabla}\times\vec{B} = \mu_0(\vec{j} + \varepsilon_0\partial_t\vec{E}) = 0 \quad\Rightarrow\quad \vec{B} = -\vec{\nabla}\psi(\vec{r})$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad\Rightarrow\quad \vec{\nabla}^2\psi(\vec{r}) = 0$$



(x=0,y=0) is the beam's design curve

For finite fields on the design curve,
Ψ can be power expanded in x and y: $\psi(x,y,z) = \sum_{n,m=0}^{\infty} b_{nm}(z)x^n y^m$

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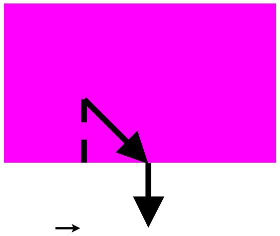
Surfaces of Equal Potential

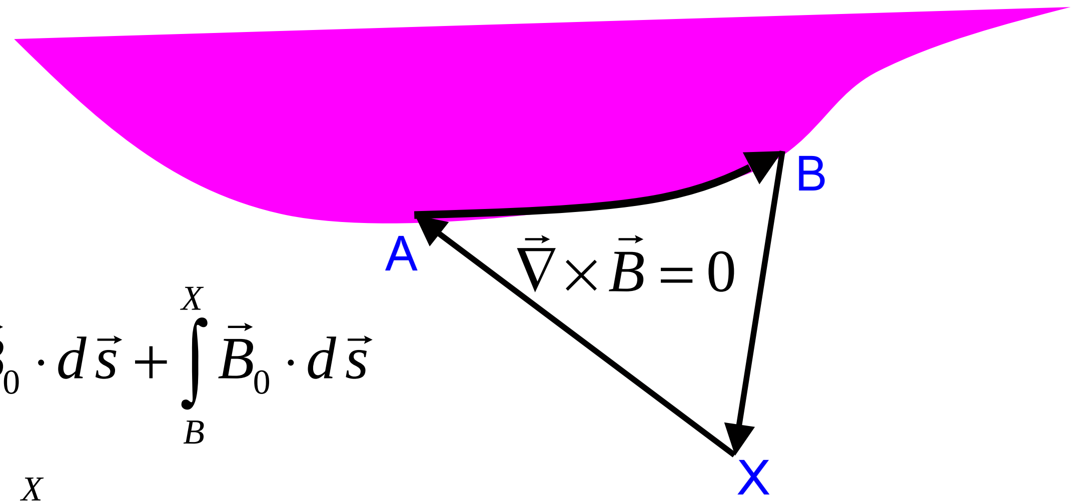


$$\vec{B}_\perp(\text{out}) = \vec{B}_\perp(\text{in})$$

$$\vec{H}_{\text{parallel}}(\text{out}) = \vec{H}_{\text{parallel}}(\text{in})$$

$$\vec{B}_{\text{parallel}}(\text{out}) = \frac{1}{\mu_r}\vec{B}_{\text{parallel}}(\text{in})$$

For large permeability, H(out) is perpendicular to the surface.



$$0 = \oint \vec{B}\cdot d\vec{s} = \int_X^A \vec{B}_0\cdot d\vec{s} + \int_A^B \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s}$$

$$= \int_X^A \vec{B}_0\cdot d\vec{s} + \frac{1}{\mu_r}\int_A^B \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s}$$

$$\approx \int_X^A \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s} = \Psi(A) - \Psi(B)$$

For highly permeable materials (like iron) surfaces have a constant potential.

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Green's Theorem

$$\vec{\nabla}^2 \psi = 0$$

Green function:

$$\vec{\nabla}_0^2 G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0)$$

$$\begin{aligned}
\psi(\vec{r}) &= \int_V \psi(\vec{r}_0)\delta(\vec{r} - \vec{r}_0)\, d^3\vec{r}_0 \\
&= \int_V \left[\psi(\vec{r}_0)\vec{\nabla}_0^2 G - G\vec{\nabla}_0^2 \psi(\vec{r}_0)\right] d^3\vec{r}_0 \\
&= \int_V \vec{\nabla}_0 \left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0 \psi(\vec{r}_0)\right] d^3\vec{r}_0 \\
&= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0 \psi(\vec{r}_0)\right] \cdot d^2\vec{r}_0 \\
&= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G + \vec{B}(\vec{r}_0) G\right] \cdot d^2\vec{r}_0
\end{aligned}$$

Knowledge of the field and the scalar magnetic potential on a closed surface inside a magnet determines the magnetic field for the complete volume which is enclosed.

Potential Expansion

If field data in a plane (for example the midplane of a cyclotron or of a beam line magnet) is known, the complete filed is determined:

$$\psi(x,y,z)=\sum_{n=0}^{\infty} b_n(x,z)\,y^n \quad \Rightarrow \quad \vec{B}(x,0,z)=-\begin{pmatrix}\partial_x b_0(x,z)\\ b_1(x,z)\\ \partial_z b_0(x,z)\end{pmatrix}$$

$$0=\vec{\nabla}^2\psi=\sum_{n=0}^{\infty}(\partial_x^2+\partial_y^2)\,b_n\,y^n+\sum_{n=2}^{\infty}n(n-1)\,b_n\,y^{n-2}$$

$$=\sum_{n=0}^{\infty}\left[(\partial_x^2+\partial_y^2)\,b_n+(n+2)(n+1)\,b_n\right]y^n$$

$$b_{n+2}(x,z)=-\frac{1}{(n+2)(n+1)}(\partial_x^2+\partial_y^2)\,b_n(x,z)$$

Data of the magnetic field in the plane y=0 is used to determine $b_0(x,z)$ and $b_1(x,z)$.

Complex Potentials

$$w = x + iy \quad , \quad \overline{w} = x - iy$$

$$\partial_x = \partial_w + \partial_{\overline{w}} \ , \quad \partial_y = i\partial_w - i\partial_{\overline{w}} = i(\partial_w - \partial_{\overline{w}})$$

$$\vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = (\partial_w + \partial_{\overline{w}}) - (\partial_w - \partial_{\overline{w}}) + \partial_z^2 = 4\partial_w\partial_{\overline{w}} + \partial_z^2$$

$$\psi = \mathrm{Im}\{ \sum_{\nu,\lambda=0}^{\infty} a_{\nu\lambda}(z) \cdot (w\overline{w})^{\lambda}\, \overline{w}^{\nu} \}$$

$$\vec{\nabla}^2\psi = \mathrm{Im}\{ \sum_{\nu=0,\lambda=1}^{\infty} 4a_{\nu\lambda}(\lambda+\nu)\lambda(w\overline{w})^{\lambda-1}\overline{w}^{\nu} + \sum_{\nu=0,\lambda=0}^{\infty} a''_{\nu\lambda}(w\overline{w})^{\lambda}\overline{w}^{\nu} \}$$

$$= \mathrm{Im}\{ \sum_{\nu,\lambda=0}^{\infty} [4(\lambda+1+\nu)(\lambda+1)a_{\nu\lambda+1} + a''_{\nu\lambda}](w\overline{w})^{\lambda}\overline{w}^{\nu} \} = 0$$

Iteration equation: $a_{\nu\lambda+1} = \dfrac{-1}{4(\lambda+1+\nu)(\lambda+1)} a''_{\nu\lambda} \quad , \quad a_{\nu 0} = \Psi_\nu(z)$

The functions $\Psi_\nu(z)$ along a line determine the complete field inside a magnet.

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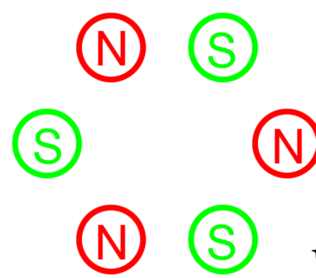
Multipole Coefficients

$\Psi_\nu(z)$ are called the z-dependent multipole coefficients

$$\psi(x,y,z) = \mathrm{Im}\{\sum_{\nu,\lambda=0}^{\infty} \frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\lambda!}\left(\frac{w\bar{w}}{4}\right)^\lambda \bar{w}^\nu \Psi_\nu^{[2\lambda]}(z)\}$$

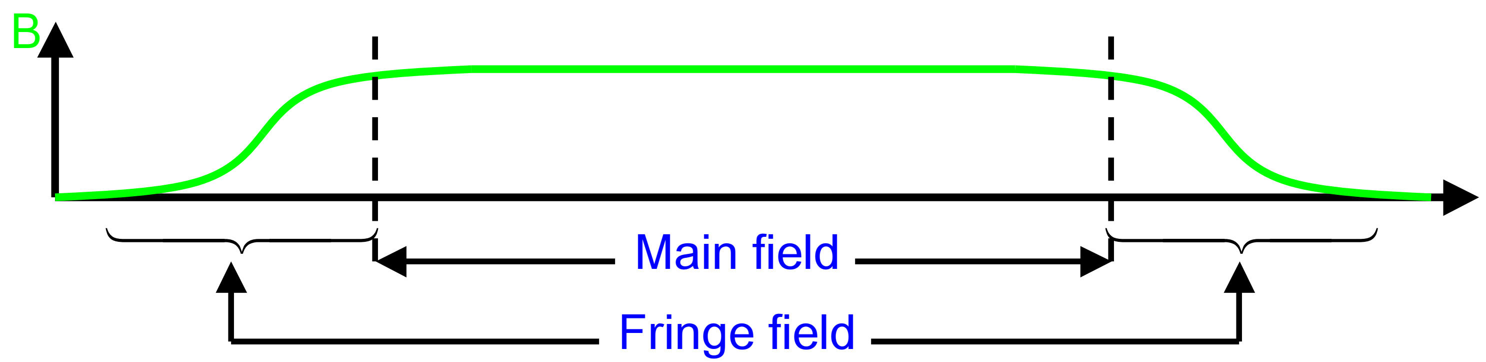
$$\psi(r,\varphi,z) = \sum_{\nu,\lambda=0}^{\infty} \frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\lambda!}\left(\frac{r}{2}\right)^{2\lambda} r^\nu \,\mathrm{Im}\{\Psi_\nu^{[2\lambda]}(z)e^{-i\nu\varphi}\}$$

The index ν describes C_ν Symmetry around the z-axis $\vec{e}_z$ due to a sign change after $\Delta\varphi = \frac{\pi}{\nu}$



$\nu = 3$

Fringe Fields and Main Fields



Only the fringe field region has terms with $\lambda \neq 0$ and $\partial_z^2 \psi \neq 0$

Main fields in accelerator physics: $\lambda = 0\,, \quad \partial_z^2 \psi = 0$

$$\Psi_\nu = \begin{cases} e^{i\nu\vartheta_\nu} |\Psi_\nu| & \text{for} \quad \nu \neq 0 \\ i \quad |\Psi_0| & \text{for} \quad \nu = 0 \end{cases}$$

$$\psi(r,\varphi) = \sum_{\nu=1}^{\infty} r^\nu |\Psi_\nu| \,\mathrm{Im}\{e^{-i\nu(\varphi-\vartheta_\nu)}\} + |\Psi_0|$$

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Main Field Potential

Main field potential: $$\psi = |\Psi_0| - \sum_{\nu=1}^{\infty} r^{\nu} |\Psi_{\nu}| \sin[\nu(\varphi - \vartheta_{\nu})]$$

The isolated multipole: $$\psi = -r^{\nu} |\Psi_{\nu}| \sin(\nu\varphi)$$

Where the rotation ϑ_{ν} of the coordinate system is set to 0

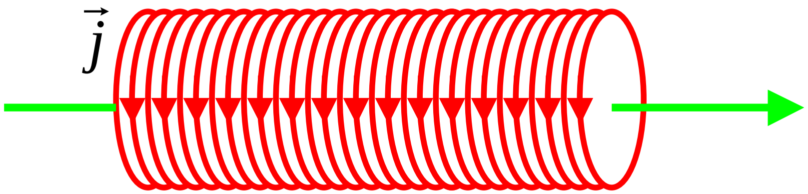
The potentials of different multipole components Ψ_{ν} have

a) Different rotation symmetry C_{ν}

b) Different radial dependence r^{ν}

Multipoles in Accelerators

ν=0: Solenoids



$$\psi = \Psi_0(z) - \frac{w\bar{w}}{4}\Psi_0''(z) \pm \dots$$

$$\vec{B} = \begin{pmatrix} \frac{x}{2}\Psi_0'' \\ \frac{y}{2}\Psi_0'' \\ -\Psi_0' \end{pmatrix} \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$m\gamma\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \begin{pmatrix} -\frac{x}{2}B_z' \\ -\frac{y}{2}B_z' \\ B_z \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{qB_z}{m\gamma}\begin{pmatrix} \dot{y} \\ -\dot{x} \end{pmatrix} + \frac{qB_z'\dot{z}}{2m\gamma}\begin{pmatrix} y \\ -x \end{pmatrix}$$

$$\Downarrow$$

$$\ddot{w} = -i\frac{qB_z}{m\gamma}\dot{w} - i\frac{q\dot{B}_z}{2m\gamma}w$$

$$g = \frac{qB_z}{2m\gamma} \quad , \quad w_0 = w\, e^{i\int_0^t g\,dt}$$

$$\ddot{w}_0 = (\ddot{w} + i2g\dot{w} + i\dot{g}w - g^2 w)\, e^{i\int_0^t g\,dt}$$

$$= -g^2 w_0$$

$$\ddot{x}_0 = -g^2 x_0$$

$$\ddot{y}_0 = -g^2 y_0$$

Focusing in a rotating coordinate system

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Solenoid Focusing

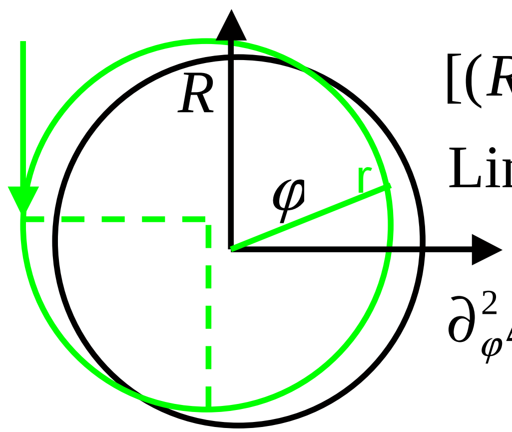
Solenoid magnets are used in detectors for particle identification via $\rho = \frac{p}{qB}$

The solenoid's rotation $\dot{\varphi} = -\frac{qB_z}{2m\gamma}$ of the beam is often compensated by a reversed solenoid called compensator.

Solenoid or Weak Focusing:

Solenoids are also used to focus low γ beams: $\ddot{w} = -\left(\frac{qB_z}{2m\gamma}\right)^2 w$

Weak focusing from natural ring focusing:



$$\Delta r = r - R$$

$$[(R+\Delta r)\cos\varphi - \Delta x_0]^2 + [(R+\Delta r)\sin\varphi - \Delta y_0]^2 = R^2$$

Linearization in Δ: $\Delta r = (\cos\varphi\,\Delta x_0 + \sin\varphi\,\Delta y_0)$

$$\partial_\varphi^2 \Delta r = -\Delta r \quad \Rightarrow \quad \Delta\ddot{r} = -\dot{\varphi}^2 \Delta r = -\left(\frac{v}{\rho}\right)^2 \Delta r = -\left(\frac{qB}{m\gamma}\right)^2 \Delta r$$

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Solenoid vs. Strong Focusing

If the solenoids field was perpendicular to the particle's motion,

its bending radius would be $\rho_z = \dfrac{p}{qB_z}$

$$\ddot{r} = -\left(\frac{qB_z}{2m\gamma}\right)^2 r = -\frac{qv_z}{m\gamma} B_z \frac{r}{4\rho_z}$$

Solenoid focusing is weak compared to the deflections created by a transverse magnetic field.

Transverse fields: $\vec{B} = B_x\,\vec{e}_x + B_y\,\vec{e}_y$

$$m\gamma \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \begin{pmatrix} B_x \\ B_y \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{qv_z}{m\gamma} \begin{pmatrix} -B_y \\ B_x \end{pmatrix}$$ Strong focusing

Weak focusing < Strong focusing by about r/ρ

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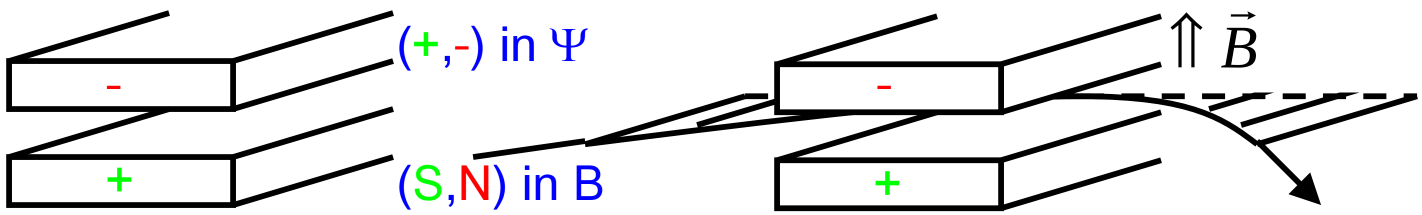
Multipoles in Accelerators

ν=1: Dipoles

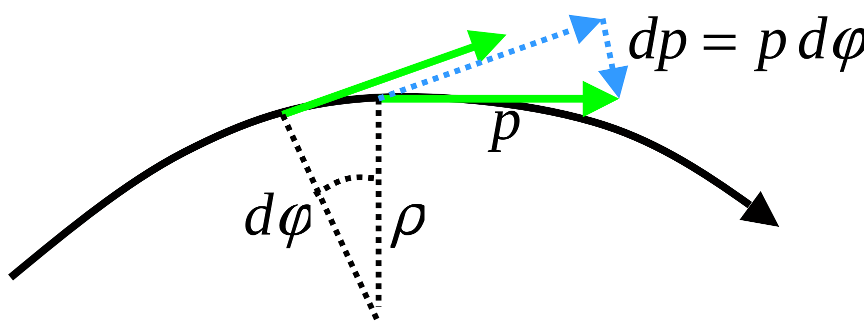
$$\psi = \Psi_1 \operatorname{Im}\{x - iy\} = -\Psi_1 \cdot y \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_1 \, \vec{e}_y$$

Equipotential
$y = \text{const.}$

C_1 Symmetry



Dipole magnets are used for steering the beams direction



$$\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B} \quad \Rightarrow \quad \frac{dp}{dt} = qvB_\perp \quad \Rightarrow \quad \rho = \frac{dl}{d\varphi} = \frac{vdt}{dp/p} = \frac{p}{qB_\perp}$$

Bending radius: $\rho = \dfrac{p}{qB}$

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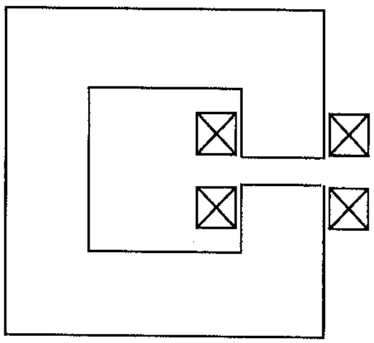
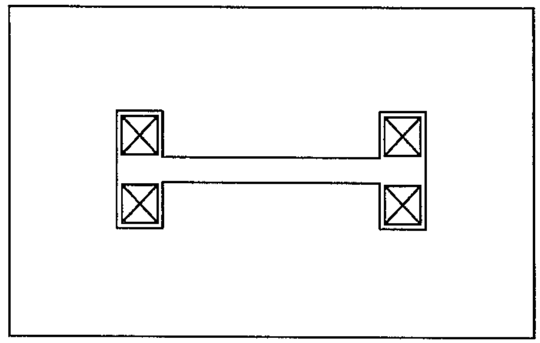
Different Dipoles

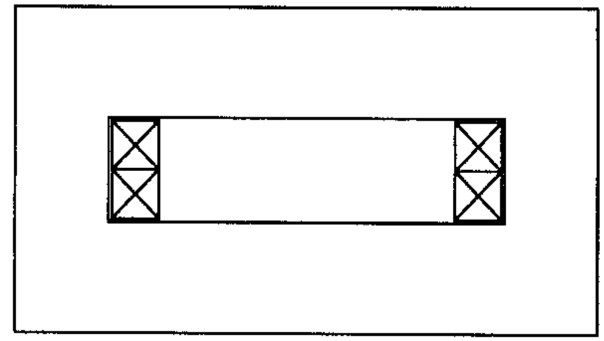
C-shape magnet:

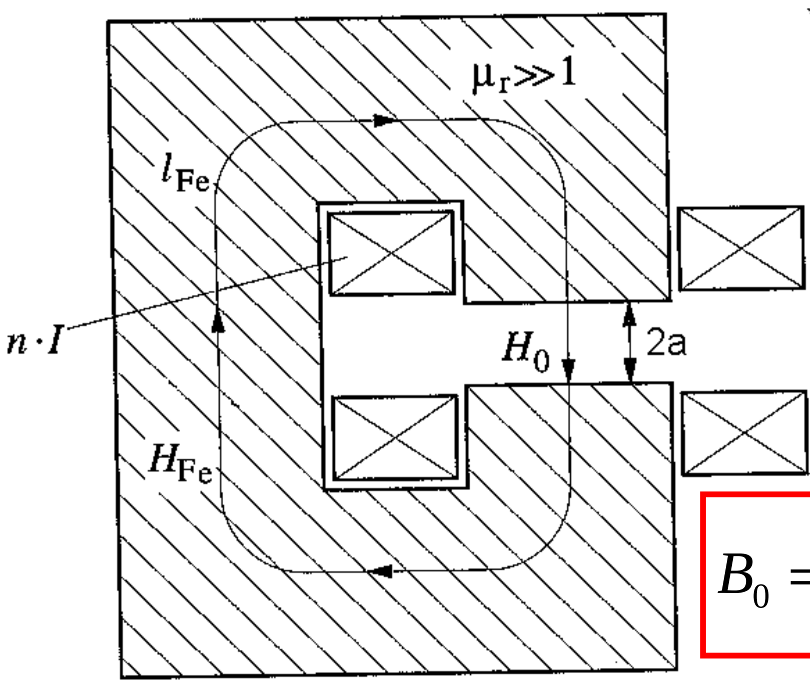


H-shape magnet:



Window frame magnet:





$$\vec{B}_\perp(\mathrm{out}) = \vec{B}_\perp(\mathrm{in})$$

$$\vec{H}_\perp(\mathrm{out}) = \mu_r \vec{H}_\perp(\mathrm{in})$$

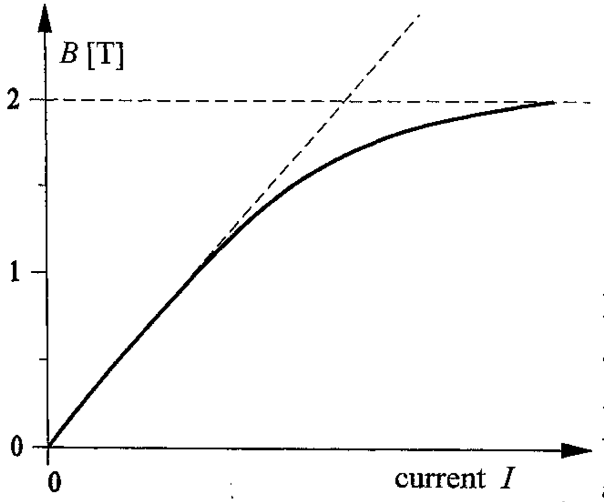
$$2nI = \oint \vec{H}\cdot d\vec{s} = H_{Fe} l_{Fe} + H_0 2a$$

$$= \tfrac{1}{\mu_r} H_0 l_{Fe} + H_0 2a \approx H_0 2a$$

$$B_0 = \mu_0 \frac{nI}{h}$$

Dipole strength: $\dfrac{1}{\rho} = \dfrac{q\mu_0}{p}\dfrac{nI}{a}$

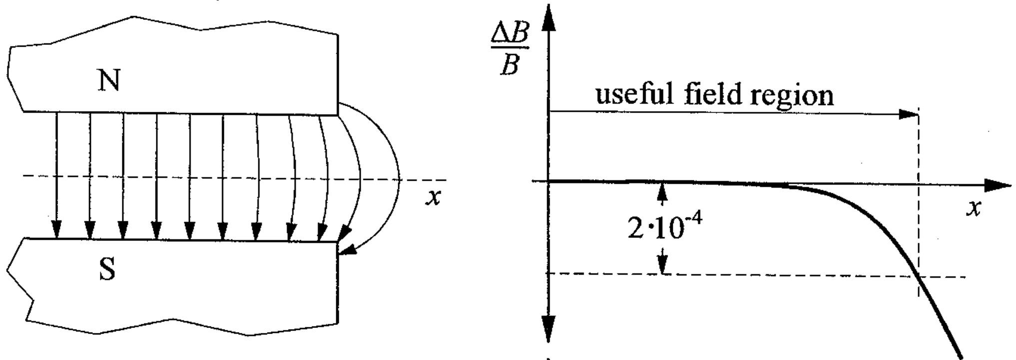
Dipole Fields

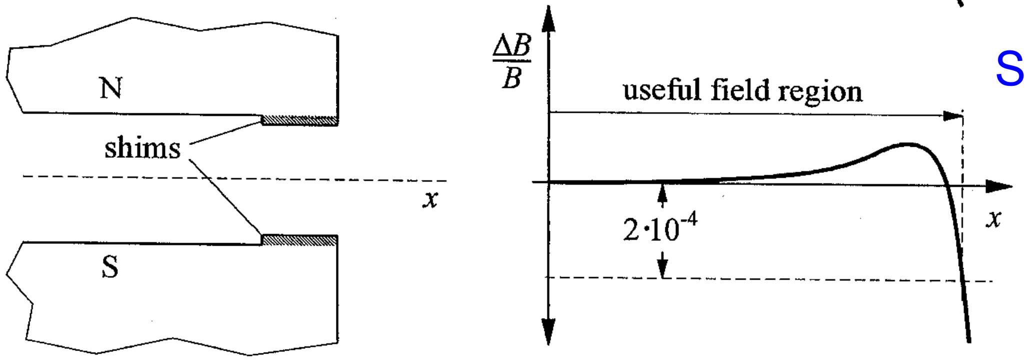


B = 2 T: Typical limit, since the field becomes dominated by the coils, not the iron. Limiting j for Cu is about 100A/mm²

B < 1.5 T: Typically used region

B < 1 T: Region in which $B_0 = \mu_0 \frac{nI}{a}$





Shims reduce the space that is open to the beam, but they also reduce the fringe field region.

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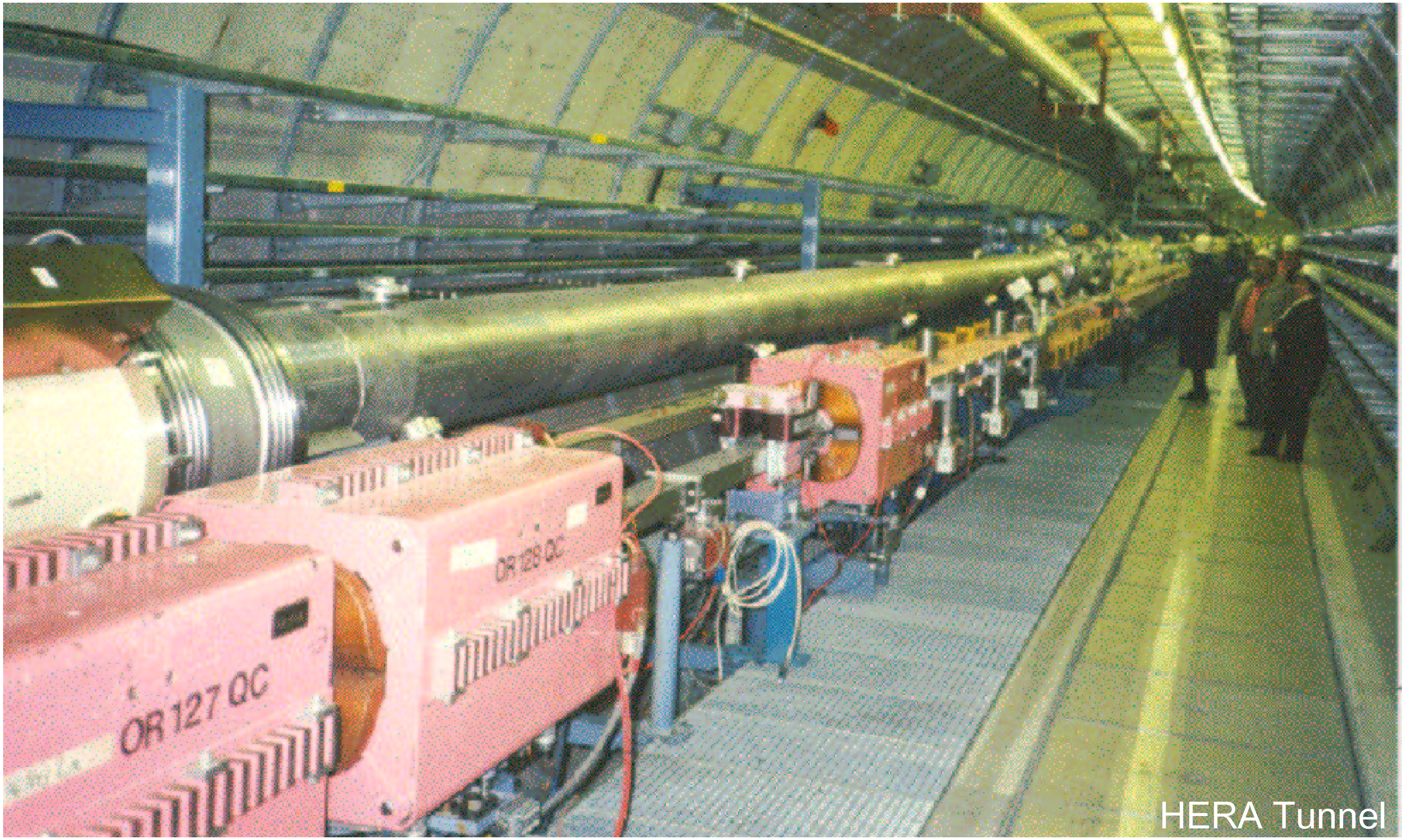
Where is the vertical Dipole?



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Exercises #1

Solutions to Homework for Physics 456/656

Introduction to Accelerator Physics and Technology (Hoffstaetter)

Due Date: Thursday, 09/11/03 - 11:40 in 311 Newman Laboratory

Exercise 1:
The main dipole magnets of the Large Electron Positron (LEP) collider had a bending radius of 3096 m.
(a) How strong was their magnetic field when LEP accelerated electrons to 100 GeV?
Answer:

$$B = \frac{p}{q\rho} = \frac{E}{cq\rho} = \frac{100 \cdot 10^9}{2.997 \cdot 10^8 3096} \frac{Vs}{m^2} = 0.1077\mathrm{T} \quad (1)$$

(b) This field strength is relatively small, why was the field not increased to increase the energy?
Answer:
Because of

$$P \propto \gamma^4 \quad (2)$$

the synchrotron radiation power would have become too large.
(c) The LEP tunnel was about 26.6km long. What fraction of it was used for bending the beam?

Answer:

$$f = \frac{2\pi\rho}{L} = 72\% \quad (3)$$

Exercise 2:
LEP produced about 20MW of synchrotron radiation when it stored electrons at 100GeV. How much would the same number of electrons have radiated at 200GeV?
Answer:

$$P(\gamma_2) = P(\gamma_1)(\frac{\gamma_2}{\gamma_1})^4 = 320MW \quad (4)$$

That would be about 30% of the output of a modern nuclear power plant, all deposited on a small stripe on the outside of the beam pipe!

Exercise 3:
Consider a storage ring built around the 40 Mm circumference of the earth, where 100% of the tunnel were used for bending particles on a circular trajectory.
(a) How large would the energy be for protons when the LHC magnets with a magnetic filed of 8.7 T were used? Could one produce the highest proton energies of the universe in this way?
Answer:

$$E = pc = \rho Bqc = \frac{4 \cdot 10^7}{2\pi} 8.7 \cdot 3.0 \cdot 10^8 eTm^2/s = 16616\mathrm{TeV} \quad (5)$$

The highest proton energies detected in the universe have more than 10^8 TeV however.
(b) How much power of synchrotron radiation would this proton beam approximately produce for the same current as in LEP (scaled from the LEP data given above)?
Answer:
The current for a given number of particles N with charge q for a ring of circumference L is given by $I = Nqc/L$. The power radiated by this current when the bending radius of the magnets is ρ is given by

$$P = \frac{c}{6\pi\epsilon_0} N \frac{q^2}{\rho^2}\gamma^4 = \frac{L}{6\pi\epsilon_0}\frac{qI}{\rho^2}\gamma^4 \,. \quad (6)$$

$$P_2 = P_1 \frac{L_2}{L_1}(\frac{\rho_1}{\rho_2})^2(\frac{m_1}{m_2})^4(\frac{E_2}{E^1})^4\frac{I_2}{I_1} = 20 \cdot 10^6 \frac{1}{26600}\frac{(2\pi \cdot 3096)^2}{4 \cdot 10^7}\frac{1}{1835^4}(\frac{16616}{0.1})^4 \mathrm{W} = 478\mathrm{GW} \quad (7)$$

(c) How large would the electron energy in this tunnel be if its synchrotron radiation load per length of the tunnel should be the same as that in LEP when the same current is stored (scaled from the LEP data given above)?
Answer:

$$\frac{P_2}{L_2} = \frac{P_1}{L_1}(\frac{\gamma_2}{\gamma_1})^4(\frac{\rho_1}{\rho_2})^2\frac{I_2}{I_1} \,, \quad (8)$$

with $P_2/L_2 = P_1/L_1$ this leads to

$$\gamma_2 = \gamma_1(\frac{\rho_2}{\rho_1})^{\frac{1}{2}} = 100GeV(\frac{4 \cdot 10^4}{2\pi \cdot 3096})^{\frac{1}{2}} = 4.53TeV \,. \quad (9)$$

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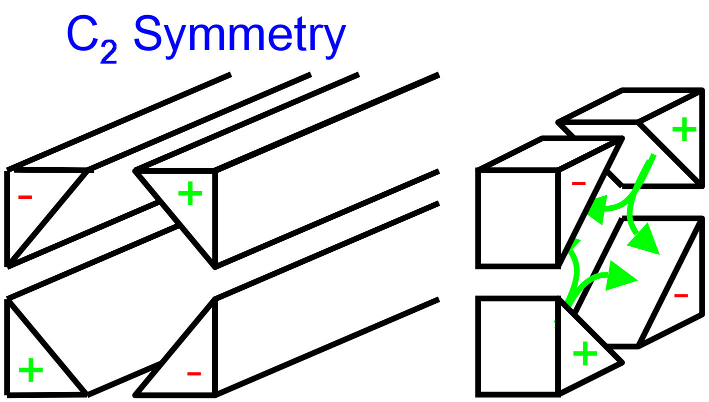
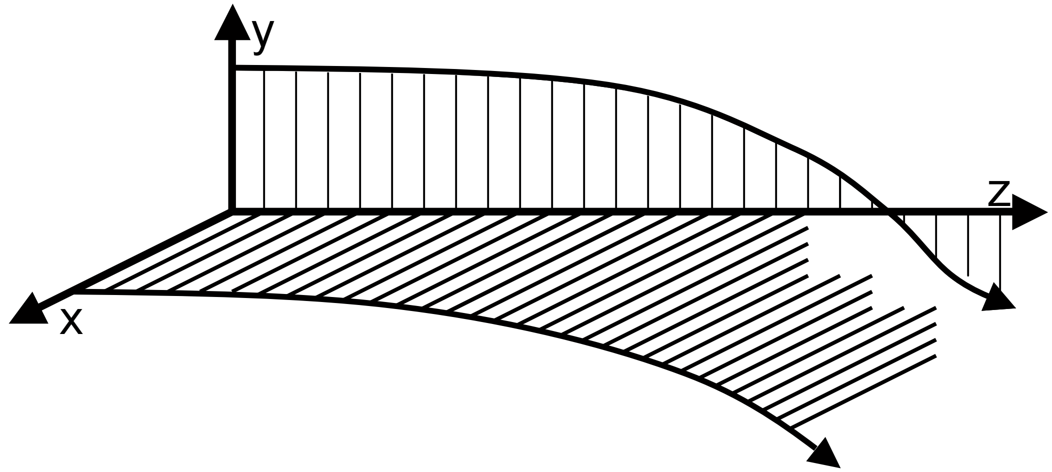
Multipoles in Accelerators
ν=2: Quadrupoles

$$\psi = \Psi_2 \,\mathrm{Im}\{(x-iy)^2\} = -\Psi_2 \cdot 2xy \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_2 \, 2\begin{pmatrix} y \\ x \end{pmatrix}$$





In a quadrupole particles are focused in one plane and defocused in the other plane. Other modes of strong focusing are not possible.

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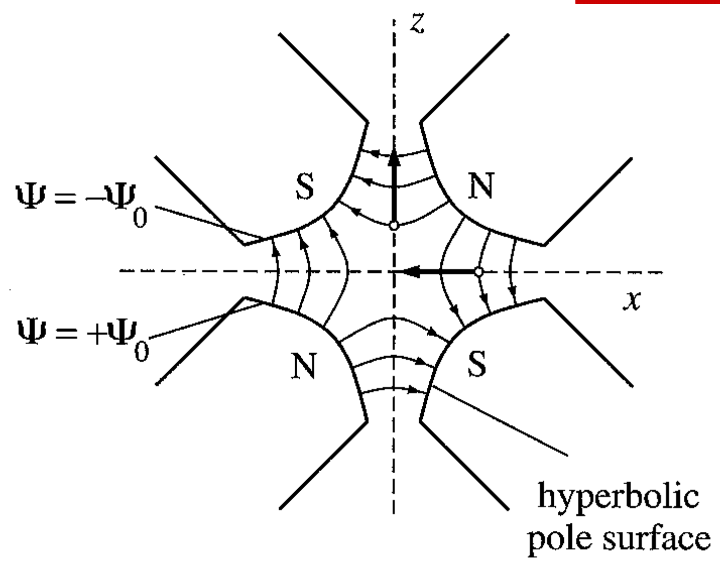
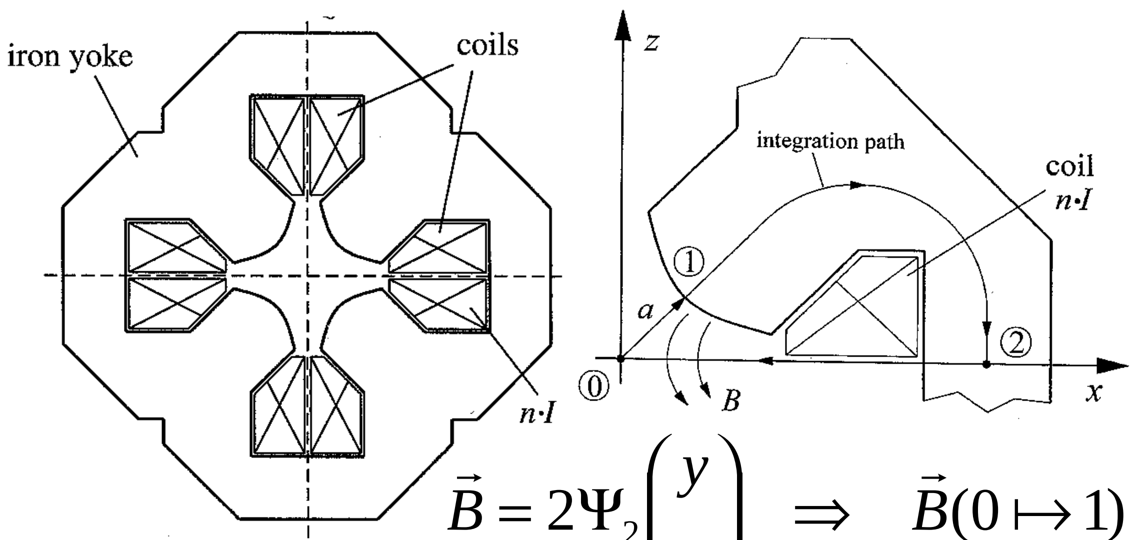
Quadrupole Fields

$$\psi = -\Psi_2 \cdot 2xy \quad \Rightarrow \quad \text{Equipotential: } x = \frac{\text{const.}}{y}$$





$$\vec{B} = 2\Psi_2 \begin{pmatrix} y \\ x \end{pmatrix} \quad \Rightarrow \quad \vec{B}(0 \mapsto 1) = 2\Psi_2 r\, \vec{e}_r$$

Quadrupole strength:

$$nI = \oint \vec{H} \cdot d\vec{s} \approx \int_0^a H_r dr = \Psi_2 \frac{a^2}{\mu_0}$$

$$k_1 = \frac{q}{p} \partial_x B_y \Big|_0 = \frac{q\mu_0}{p} \frac{2nI}{a^2}$$

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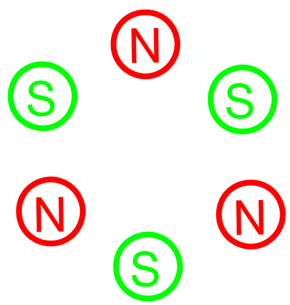
Real Quadrupoles

The coils show that this is an upright quadrupole not a rotated or skew quadrupole.

Multipoles in Accelerators
ν=3: Sextupoles (Hexapoles)

$$\psi = \Psi_3 \,\mathrm{Im}\{(x-iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

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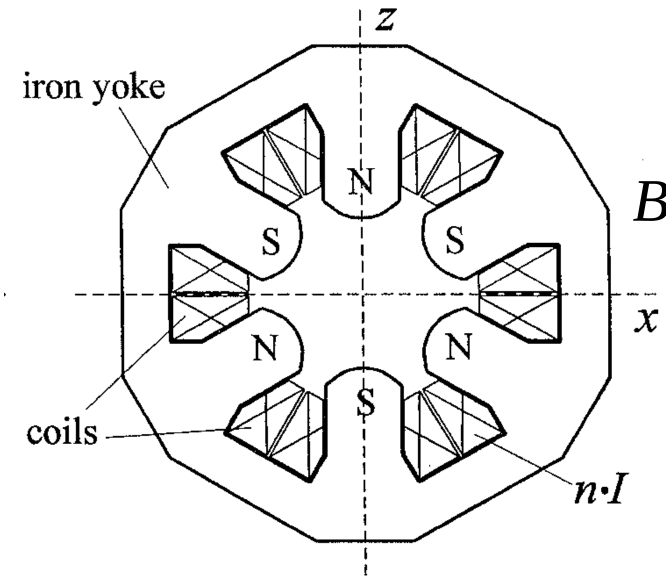
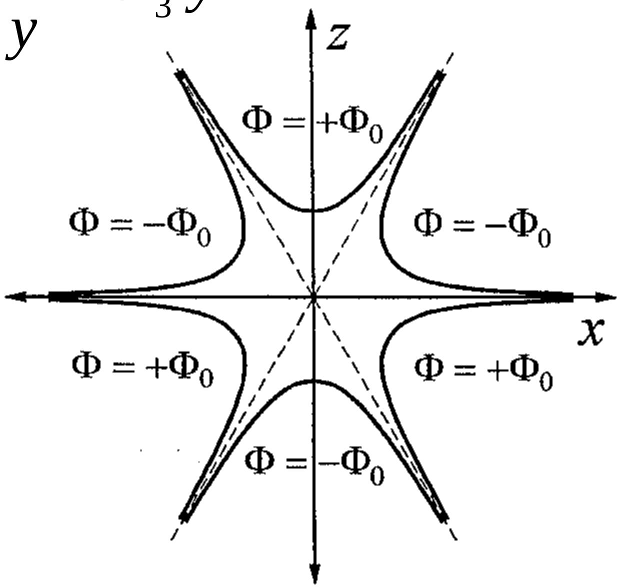
Sextupole Fields

$$\psi = \Psi_2 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \text{Equipotential: } x = \sqrt{\frac{\text{const.}}{y} + \tfrac{1}{3} y^2}$$



$$B_y\big|_{x=0} = -\Psi_3 3y^2$$



$$nI = \oint \vec{H} \cdot d\vec{s} \approx \int_0^a H_r dr = \Psi_3 \frac{a^3}{\mu_0}$$

Quadrupole strength:

$$k_2 = \frac{q}{p} \partial_x^2 B_y\big|_0 = \frac{q\mu_0}{p} \frac{6nI}{a^3}$$

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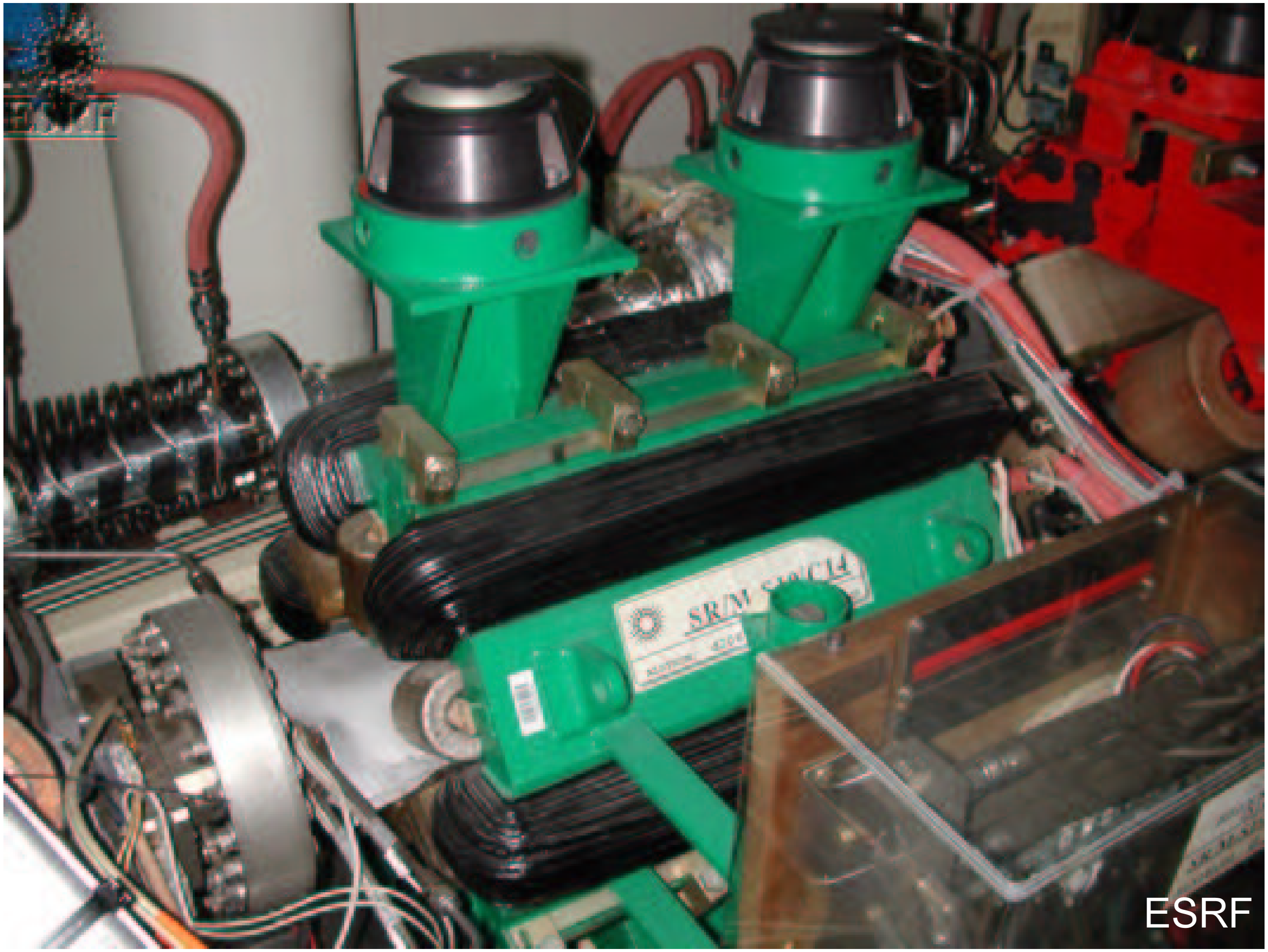
Real Sextupoles



The CESR Tunnel

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Higher order Multipoles

$$\psi = \Psi_n \,\mathrm{Im}\{(x - iy)^n\} = \Psi_n \cdot (\ldots - i\,n\,x^{n-1} y) \quad \Rightarrow \quad \vec{B}(y=0) = \Psi_n\, n \begin{pmatrix} 0 \\ x^{n-1} \end{pmatrix}$$

Multipole strength: $k_n = \frac{q}{p} \partial_x^n B_y \Big|_{x,y=0} = \frac{q}{p} \Psi_{n+1}\,(n+1)!$ units: $\frac{1}{\mathrm{m}^{n+1}}$

p/q is also called Bρ and used to describe the energy of multiply charge ions

Names: dipole, quadrupole, sextupole, octupole, decapole, duodecapole, …

Higher order multipoles come from

- Field errors in magnets
- Magnetized materials
- From multipole magnets that compensate such erroneous fields
- To compensate nonlinear effects of other magnets
- To stabilize the motion of many particle systems
- To stabilize the nonlinear motion of individual particles

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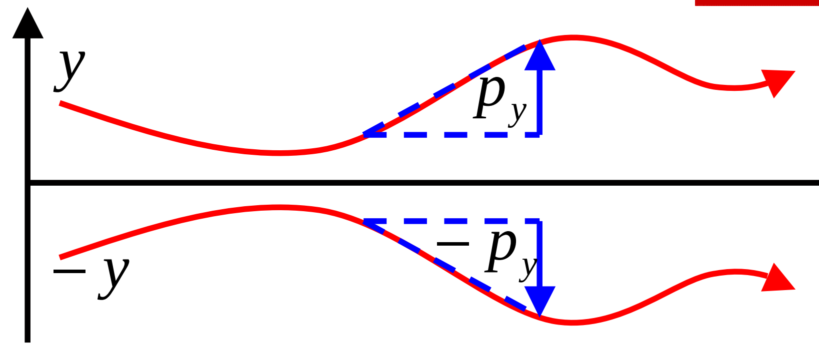
Midplane Symmetric Motion

$$\vec{r}^{\oplus} = (x, -y, z)$$

$$\vec{p}^{\oplus} = (p_x, -p_y, p_z)$$

$$\tfrac{d}{dt}\vec{p} = \vec{F}(\vec{r}, \vec{p}) \quad \Rightarrow \quad \tfrac{d}{dt}\vec{p}^{\oplus} = \vec{F}(\vec{r}^{\oplus}, \vec{p}^{\oplus})$$



$$\begin{aligned}
v_y B_z - v_z B_y &= -v_y B_z(x,-y,z) - v_z B_y(x,-y,z) & B_x(x,-y,z) &= -B_x(x,y,z)\\
v_z B_x - v_x B_z &= v_z B_x(x,-y,z) - v_x B_z(x,-y,z) \Rightarrow & B_y(x,-y,z) &= B_y(x,y,z)\\
v_x B_y - v_y B_x &= v_x B_y(x,-y,z) + v_y B_x(x,-y,z) & B_z(x,-y,z) &= -B_z(x,y,z)
\end{aligned}$$

$$\boxed{\psi(x,-y,z) = -\psi(x,y,z)}$$

$$\Psi_n \,\mathrm{Im}\left\{e^{in\vartheta_n}(x+iy)^n\right\} = -\Psi_n \,\mathrm{Im}\left\{e^{in\vartheta_n}(x+iy)^n\right\}$$

$$\Rightarrow \quad \Psi_n \,\mathrm{Im}\left[e^{in\vartheta_n}\, 2\,\mathrm{Re}\left\{(x+iy)^n\right\}\right] = 0 \quad \Rightarrow \quad \boxed{\vartheta_n = 0}$$

The discussed multipoles

produce midplane symmetric motion. When the field is rotated by $\pi/2$, i.e $\vartheta_n = \pi/2n$, one speaks of a skew multipole.

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Superconducting Magnets

Above 2T the field from the bare coils dominate over the magnetization of the iron.

But Cu wires cannot create much filed without iron poles:

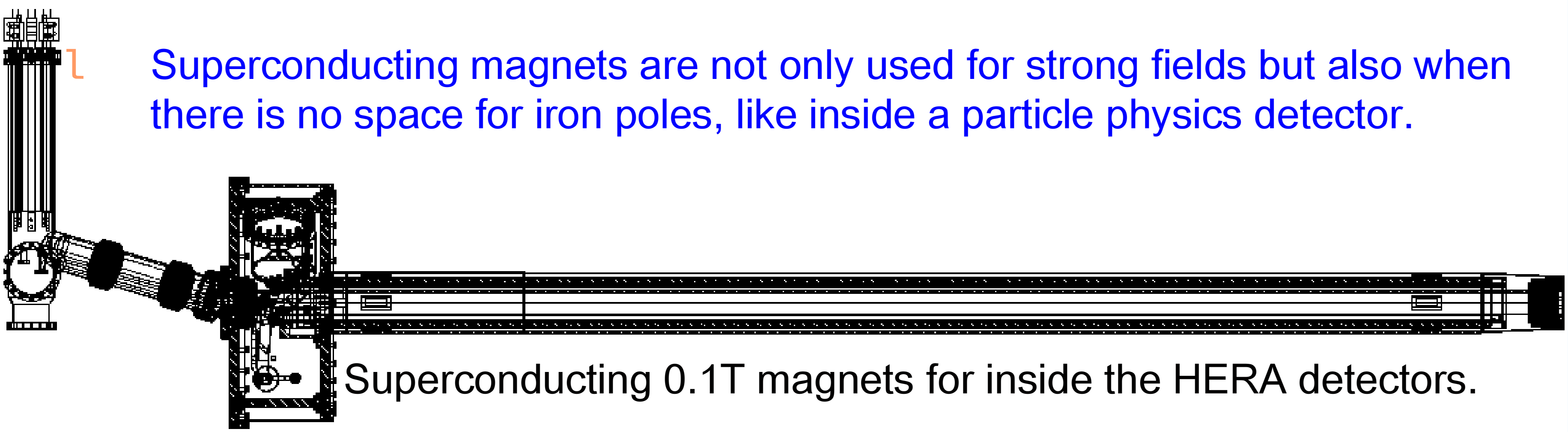
5T at 5cm distance from a 3cm wire would require a current density of

$$j = \frac{I}{d^2} = \frac{1}{d^2}\frac{2\pi\ r\,B}{\mu_0} = 1389\frac{\mathrm{A}}{\mathrm{mm}^2}$$

Cu can only support about 100A/mm^2.

- Superconducting cables routinely allow current densities of 1500A/mm^2 at 4.6 K and 6T. Materials used are usually Nb aloys, e.g. NbTi, Nb_3Ti or Nb_3Sn.

- Superconducting magnets are not only used for strong fields but also when there is no space for iron poles, like inside a particle physics detector.



Superconducting 0.1T magnets for inside the HERA detectors.

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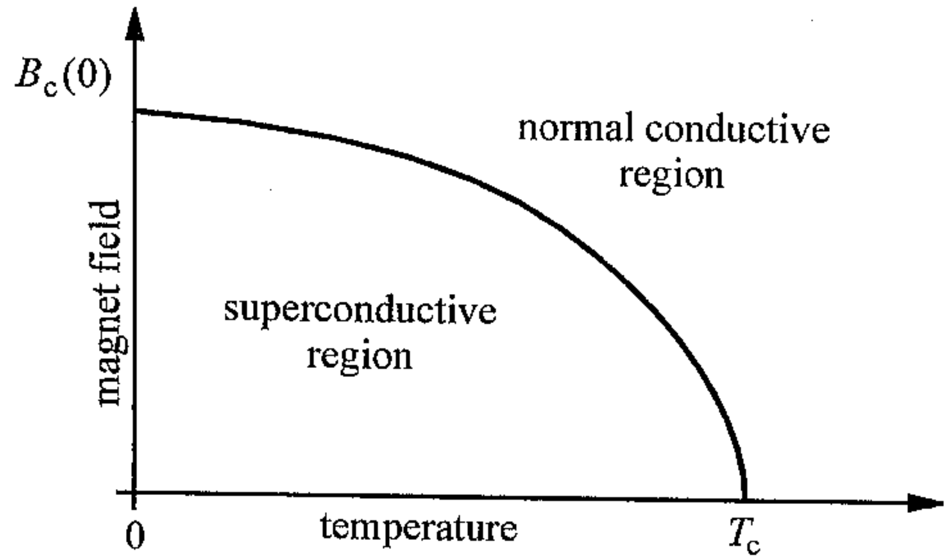
Superconducting Magnets

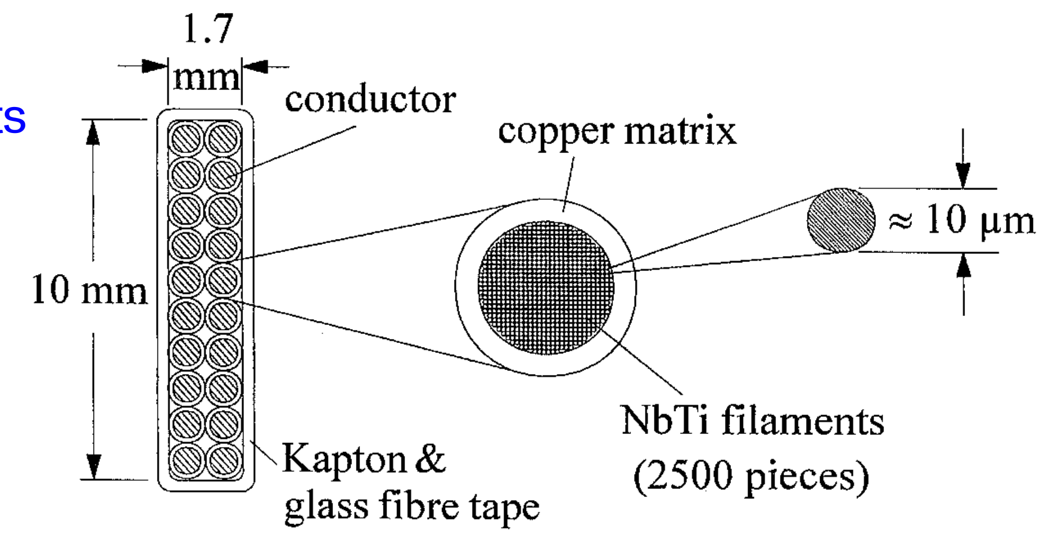
Problems:

- Superconductivity brakes down for too large fields
- Due to the Meissner-Ochsenfeld effect superconductivity current only flows on a thin surface layer.



Remedy:

- Superconducting cable consists of many very thin filaments (about 10µm).



Complex Potential of a Wire

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Straight wire at the origin: $\vec{\nabla}\times\vec{B}=\mu_0\,\vec{j} \quad\Rightarrow\quad \vec{B}(r)=\frac{\mu_0\,I}{2\pi\,r}\vec{e}_\varphi$

Wire at $\vec{a}$:

$$\vec{B}(x,y)=\frac{\mu_0\,I}{2\pi\,(\vec{r}-\vec{a})^2}\begin{pmatrix}-[y-a_y]\\ x-a_x\end{pmatrix}=\frac{\mu_0 I}{2\pi}\frac{1}{a^2+r^2-2ar\cos(\varphi-\varphi_a)}\begin{pmatrix}-[y-a_y]\\ x-a_x\end{pmatrix}$$

This can be represented by complex multipole coefficients Ψ_ν

$$\vec{B}(x,y)=-\vec{\nabla}\Psi \quad\Rightarrow\quad B_x+iB_y=-(\partial_x+i\partial_y)\psi=-2\partial_{\overline{w}}\psi$$

$$B_x+iB_y=\frac{\mu_0 I}{2\pi}\frac{-i(w_a-w)}{(w_a-w)(\overline{w}_a-\overline{w})}=\frac{\mu_0 I}{2\pi}\frac{-i\,\frac{w_a}{a^2}}{1-\frac{\overline{w}w_a}{a^2}}$$

$$=i\frac{\mu_0 I}{2\pi}\partial_{\overline{w}}\ln(1-\tfrac{\overline{w}w_a}{a^2})=-2\partial_{\overline{w}}\,\mathrm{Im}\left\{\frac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}$$

$$\psi=\mathrm{Im}\left\{\frac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}=-\mathrm{Im}\left\{\frac{\mu_0 I}{2\pi}\sum_{\nu=1}^{\infty}\frac{1}{\nu}\left(\frac{w_a}{a^2}\right)^\nu\overline{w}^\nu\right\} \quad\Rightarrow\quad \Psi_\nu=\frac{\mu_0 I}{2\pi}\frac{1}{\nu}\frac{1}{a^\nu}e^{i\nu\varphi_a}$$

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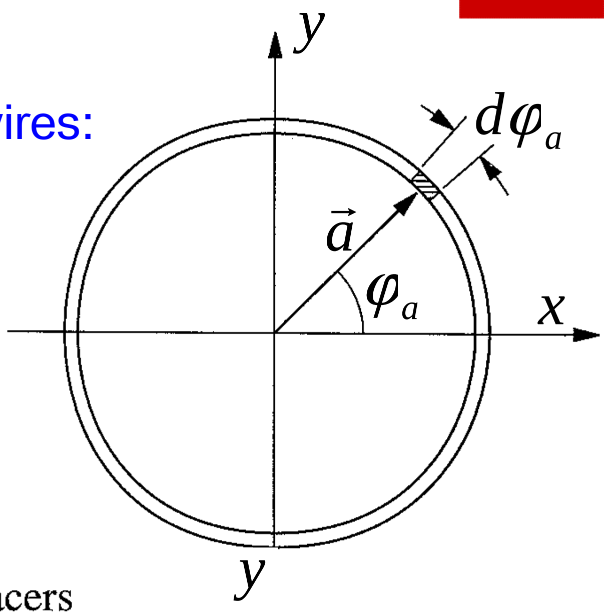
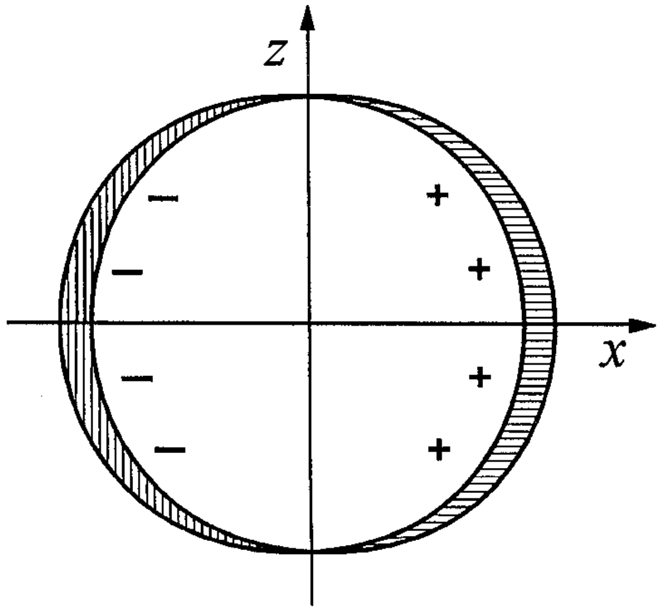
Air-coil Multipoles

Creating a multipole be created by an arrangement of wires:

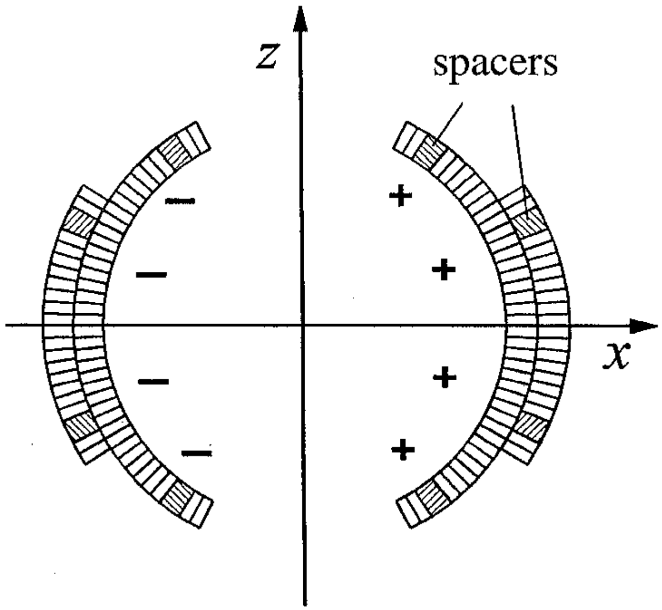
$$\Psi_\nu = \int_0^{2\pi} \frac{\mu_0}{2\pi}\frac{1}{\nu}\frac{1}{a^\nu} e^{i\nu\varphi_a} \frac{dI}{d\varphi_a} d\varphi_a$$

$$\Psi_\nu = \delta_{\nu n} \frac{\mu_0}{2}\frac{1}{n}\frac{1}{a^n}\hat{I} \quad \text{if } I(\varphi_a) = \hat{I}\cos n\varphi_a$$





Ideal multipole



Approximate multipole

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Real Air-coil Multipoles

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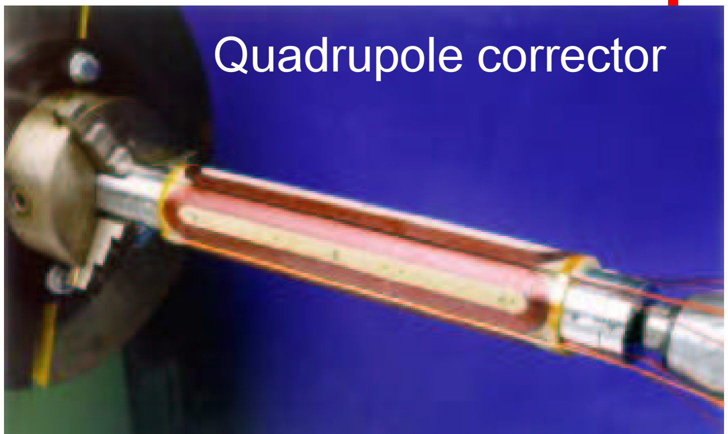
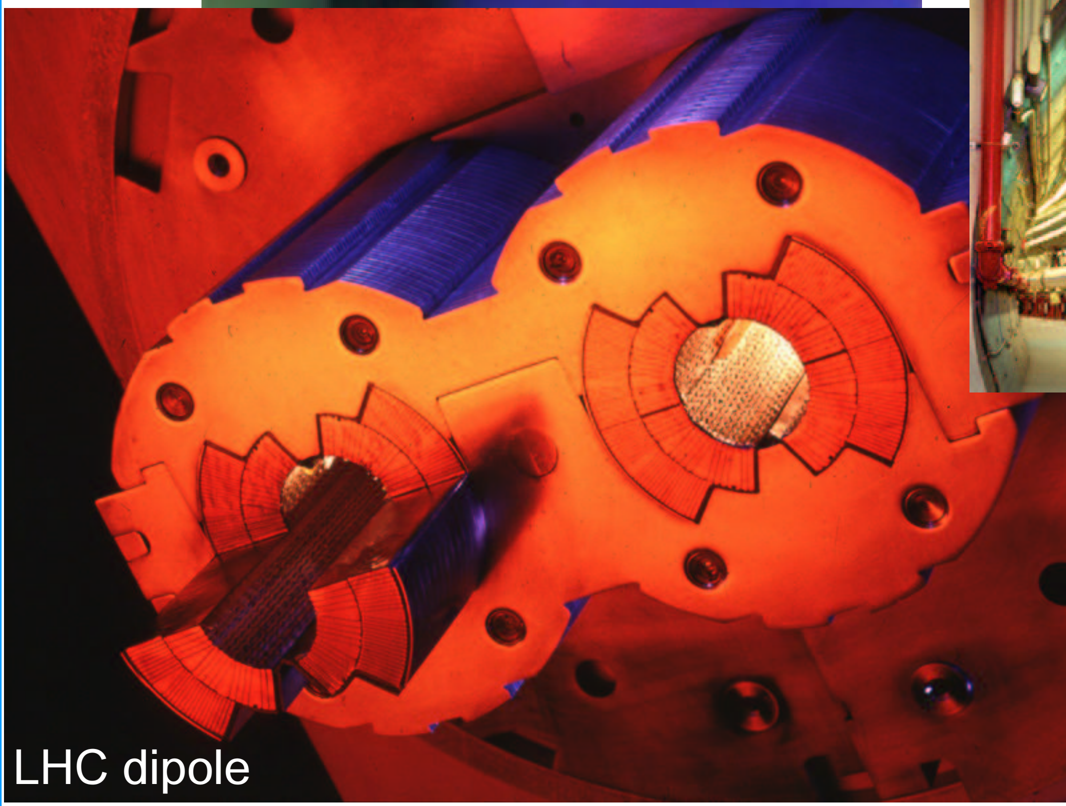

Quadrupole corrector


RHIC Tunnel


LHC dipole

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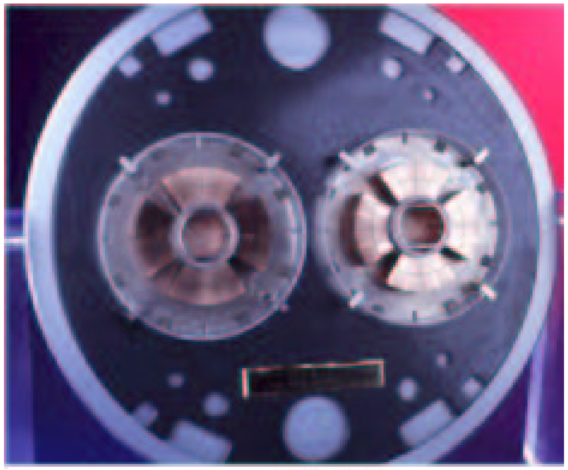
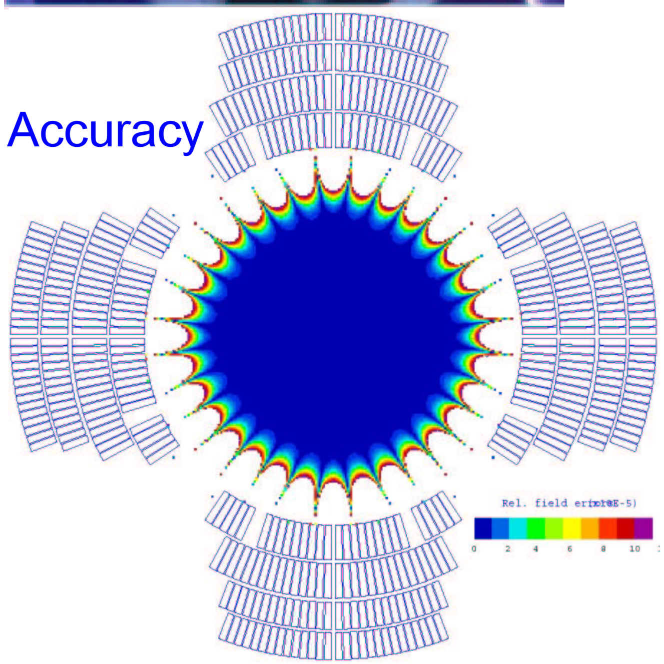
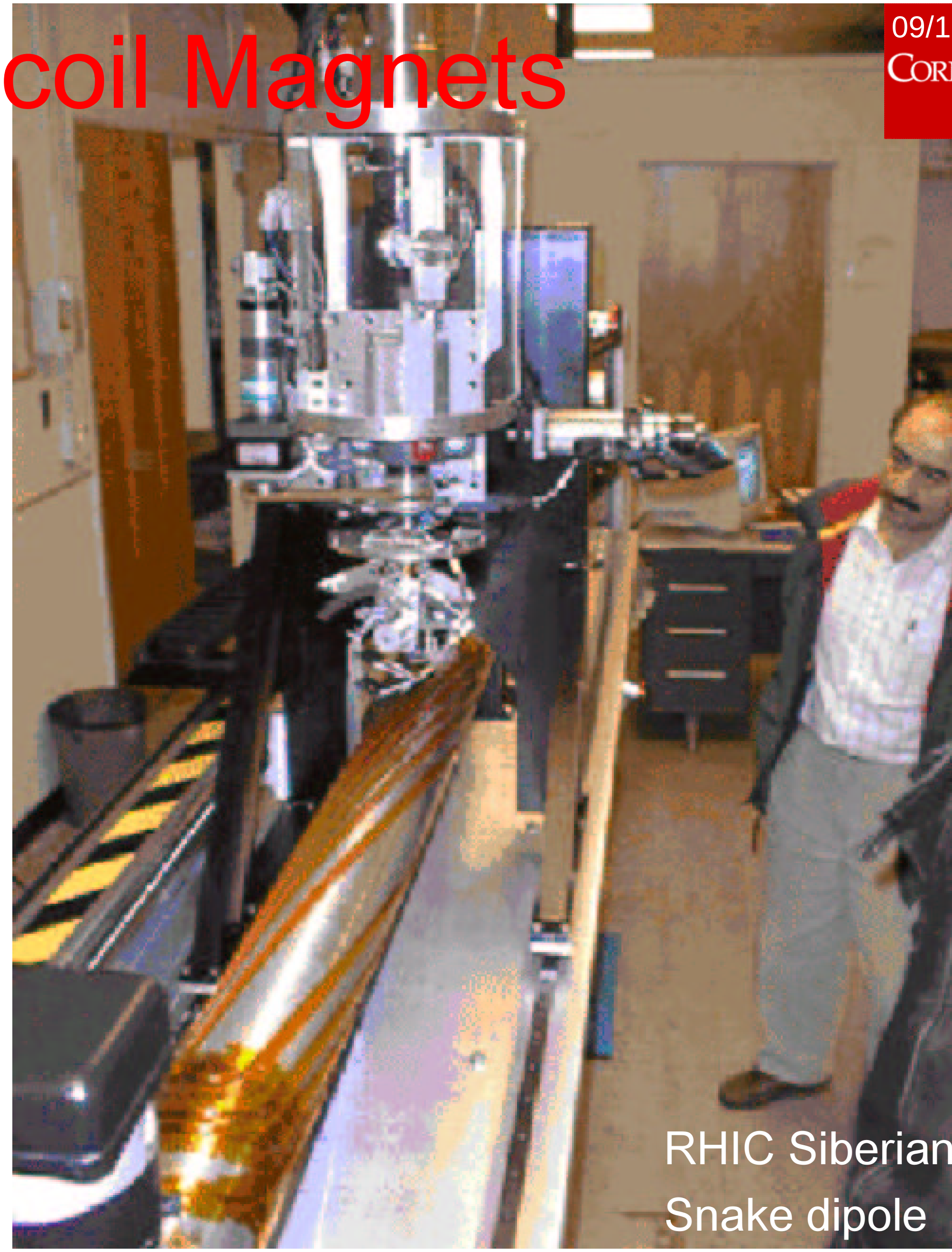
Special SC Air-coil Magnets

LHC double quadrupole



Accuracy





RHIC Siberian
Snake dipole

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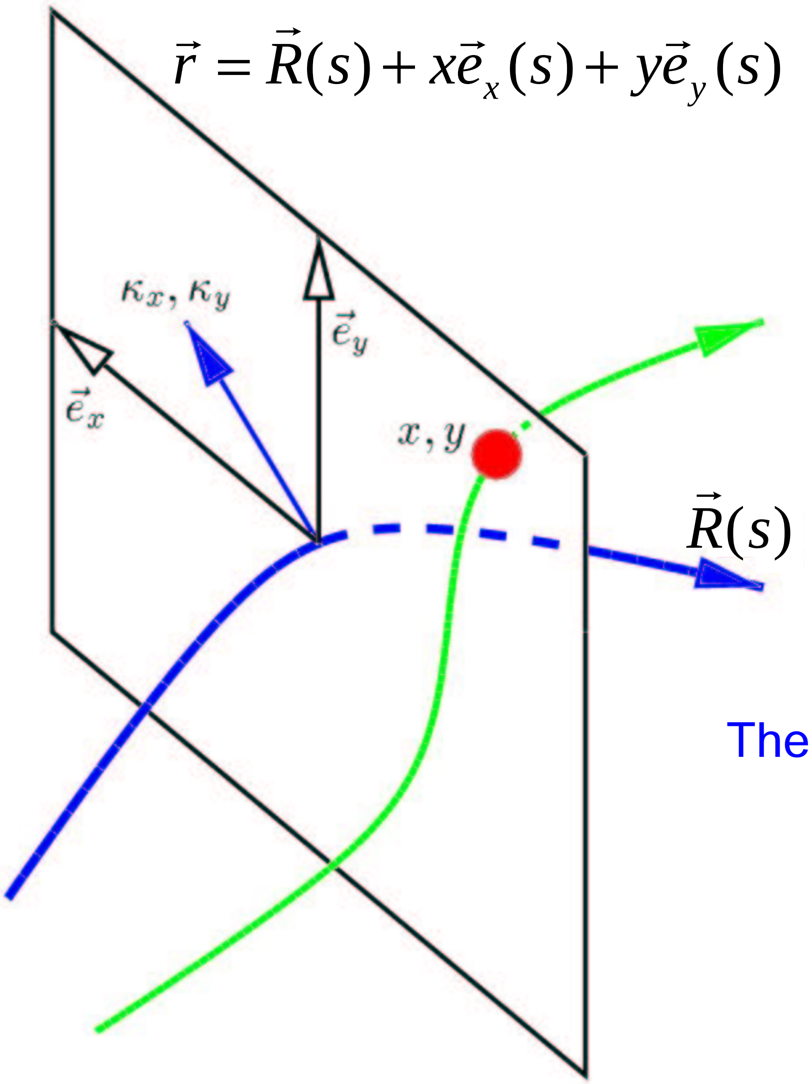
The comoving Coordinate System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$\left|d\vec{R}\right| = ds$$

$$\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)$$

The time dependence of a particle's motion is often not as interesting as the trajectory along the accelerator length "s".

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The 4D Equation of Motion

$$\frac{d^2}{dt^2}\vec{r} = \vec{f}_r(\vec{r}, \tfrac{d}{dt}\vec{r}, t)$$

3 dimensional ODE of 2nd order can be changed to a
6 dimensional ODE of 1st order:

$$\left.\begin{aligned}\frac{d}{dt}\vec{r} &= \frac{1}{m\gamma}\vec{p} = \frac{c}{\sqrt{p^2-(mc)^2}}\vec{p}\\ \frac{d}{dt}\vec{p} &= \vec{F}(\vec{r},\vec{p},t)\end{aligned}\right\} \quad \frac{d}{dt}\vec{Z} = \vec{f}_Z(\vec{Z},t)\,, \quad \vec{Z} = (\vec{r},\vec{p})$$

If the force does not depend on time, as in a typical beam line magnet, the energy is conserved so that one can reduce the dimension to 5. The equation of motion is then autonomous.

Furthermore, the time dependence is often not as interesting as the trajectory along the accelerator length “s”. Using “s” as the independent variable reduces the dimensions to 4. The equation of motion is then no longer autonomous.

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s)\,, \quad \vec{z} = (x, y, p_x, p_y)$$

The 6D Equation of Motion

Usually one prefers to compute the trajectory as a function of "s" along the accelerator even when the energy is not conserved, as when accelerating cavities are in the accelerator.

Then the energy "E" and the time "t" at which a particle arrives at the cavities are important. And the equations become 6 dimensional again:

$$\tfrac{d}{ds}\vec{z} = \vec{f}_z(\vec{z}, s)\,, \quad \vec{z} = (x, y, p_x, p_y, -t, E)$$

But: $\vec{z} = (\vec{r}, \vec{p})$ is an especially suitable variable, since it is a phase space vector so that its equation of motion comes from a Hamiltonian, or by variation principle from a Lagrangian.

$$\delta\int\left[p_x\dot{x} + p_y\dot{y} + p_s\dot{s} - H(\vec{r}, \vec{p}, t)\right]dt = 0 \quad \Rightarrow$$ Hamiltonian motion

$$\delta\int\left[p_x x' + p_y y' - H\,t' + p_s(x, y, p_x, p_y, t, H)\right]ds = 0 \quad \Rightarrow$$ Hamiltonian motion

The new canonical coordinates are: $\vec{z} = (x, y, p_x, p_y, -t, E)$ with $\mathrm{E} = \mathrm{H}$

The new Hamiltonian is: $K = -p_s(\vec{z}, s)$

Significance of Hamiltonian

The equations of motion can be determined by one function:

$$\tfrac{d}{ds}x = \partial_{p_x} H(\vec{z},s)\,,\quad \tfrac{d}{ds}p_x = -\partial_x H(\vec{z},s)\,,\quad \ldots$$

$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial} H(\vec{z},s) = \vec{F}(\vec{z},s) \quad \text{with} \quad \underline{J} = \mathrm{diag}(\underline{J}_2)\,,\quad \underline{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The force has a Hamiltonian Jacobi Matrix:

A linear force: $\vec{F}(\vec{z},s) = \underline{F}(s)\cdot\vec{z}_0$

The Jacobi Matrix of a linear force: $\underline{F}(s)$

The general Jacobi Matrix : $F_{ij} = \partial_{z_j} F_i$ or $\underline{F} = \left(\vec{\partial}\,\vec{F}^T\right)^T$

Hamiltonian Matrices: $\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$

Prove : $F_{ij} = \partial_{z_j} F_i = \partial_{z_j} J_{ik} \partial_{z_k} H = J_{ik}\,\partial_k \partial_j H \quad \Rightarrow \quad \underline{F} = \underline{J}\,\underline{DH}$

$$\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = \underline{J}\,\underline{DJH} + \underline{J}\,\underline{D}^T\,\underline{J}^T H = 0$$

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H ↦ Symplectic Flows

The flow of a Hamiltonian equation of motion has a symplectic Jacobi Matrix

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

A linear flow: $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$

The Jacobi Matrix of a linear flow: $\underline{M}(s)$

The general Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = (\vec{\partial}_0 \vec{M}^T)^T$

The Symplectic Group SP(2N) : $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\tfrac{d}{ds}\vec{z} = \tfrac{d}{ds}\vec{M}(s, \vec{z}_0) = \underline{J}\,\vec{\nabla} H = \vec{F}$$

$$\tfrac{d}{ds} M_{ij} = \partial_{z_{0j}} F_i(\vec{z}, s) = \partial_{z_{0j}} M_k \partial_{z_k} F_i(\vec{z}, s)$$

$$\tfrac{d}{ds}\underline{M}(s, \vec{z}_0) = \underline{F}(\vec{z}, s)\underline{M}(s, \vec{z}_0)$$

$$K = \underline{M}\,\underline{J}\,\underline{M}^T$$

$$\tfrac{d}{ds}\underline{K} = \tfrac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\tfrac{d}{ds}\underline{M}^T = \underline{F}\,\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\underline{M}^T\underline{F}^T = \underline{F}\,\underline{K} + \underline{K}\,\underline{F}^T$$

K = J is a solution. Since this is a linear ODE , K = J is the unique solution.

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Symplectic Flows ↦ H

For every symplectic transport map there is a Hamilton function

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

Force vector: $\vec{h}(\vec{z}, s) = -\underline{J}\left[\frac{d}{ds}\vec{M}(s, \vec{z}_0)\right]_{\vec{z}_0 = \vec{M}^{-1}(\vec{z}, s)}$

Since then: $\frac{d}{ds}\vec{z} = \underline{J}\vec{h}(\vec{z}, s)$

There is a Hamilton function H with: $\vec{h} = \vec{\partial}H$

If and only if: $\partial_{z_j} h_i = \partial_{z_i} h_j \quad \Rightarrow \quad \underline{h} = \underline{h}^T$

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J} \quad \Rightarrow \quad \begin{cases} \frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T = -\underline{M}\,\underline{J}\frac{d}{ds}\underline{M}^T \\ \underline{M}^{-1} = -\underline{J}\,\underline{M}^T\underline{J} \end{cases}$$

$$\vec{h} \circ \vec{M} = -\underline{J}\frac{d}{ds}\vec{M}$$

$$\underline{h}(\vec{M})\underline{M} = -\underline{J}\frac{d}{ds}\underline{M}$$

$$\underline{h}(\vec{M}) = -\underline{J}\frac{d}{ds}\underline{M}\,\underline{M}^{-1} = \underline{J}\frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T\underline{J} = -\underline{J}\,\underline{M}\,\underline{J}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{M}^{-T}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{h}^T$$

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Generating Functions

The motion of particles can be represented by Generating Functions

Each flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

With a Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = \left(\vec{\partial}_0 \vec{M}^T\right)^T$

That is Symplectic: $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

Can be represented by a Generating Function:

$$F_1(\vec{q}, \vec{q}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1$$

$$F_2(\vec{p}, \vec{q}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_p F_2 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_2$$

$$F_3(\vec{q}, \vec{p}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_3 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_3$$

$$F_4(\vec{p}, \vec{p}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_q F_4 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_4$$

6-dimensional motion needs only one function ! But to obtain the transport map this has to be inverted.

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F $\mapsto$ SP(2N)

Generating Functions produce symplectic tranport maps

$$F_1(\vec q,\vec q_0,s) \quad \text{with} \quad \vec p = -\vec\partial_q F_1(\vec q,\vec q_0,s) \quad , \quad \vec p_0 = \vec\partial_{q_0} F_1(\vec q,\vec q_0,s)$$

$$\left.\begin{aligned} \vec z &= \begin{pmatrix} \vec q \\ \vec p \end{pmatrix} = \begin{pmatrix} \vec q \\ -\vec\partial_q F_1(\vec q,\vec q_0,s) \end{pmatrix} = \vec f(\vec Q,s) \\ \vec z_0 &= \begin{pmatrix} \vec q_0 \\ \vec p_0 \end{pmatrix} = \begin{pmatrix} \vec q_0 \\ \vec\partial_{q_0} F_1(\vec q,\vec q_0,s) \end{pmatrix} = \vec g(\vec Q,s) \end{aligned}\right\} \quad \begin{aligned} \vec z &= \vec f(\vec g^{-1}(\vec z_0,s),s) \\ \vec M &= \vec f \circ \vec g^{-1} \end{aligned}$$

(function concatenation)

Jacobi matrix of concatenated functions:

$$\vec C(\vec z_0) = \vec A \circ \vec B(\vec z_0)$$

$$C_{ij} = \partial_j C_i = \sum_k \partial_{z_{0j}} B_k(\vec z_0)\left[\partial_{z_k} A_i(\vec z)\right]_{\vec z=\vec B(\vec z_0)} \quad \Rightarrow \quad \underline{C} = \underline{A}(\vec B)\underline{B}$$

$$\vec M \circ \vec g = \vec f \quad \Rightarrow \quad \underline{M}(\vec g) = \underline{F}\,\underline{G}^{-1}$$

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$$\vec f(\vec Q,s)=\begin{pmatrix}\vec q\\ -\vec\partial_q F_1(\vec q,\vec q_0,s)\end{pmatrix}\;\Rightarrow\; F=\begin{pmatrix}\underline{1} & \underline{0}\\ -\vec\partial_q\vec\partial_q^T F_1 & -\vec\partial_q\vec\partial_{q_0}^T F_1\end{pmatrix}$$

$$\vec g(\vec Q,s)=\begin{pmatrix}\vec q_0\\ \vec\partial_{q_0} F_1(\vec q,\vec q_0,s)\end{pmatrix}\;\Rightarrow\; G=\begin{pmatrix}\underline{0} & \underline{1}\\ \vec\partial_{q_0}\vec\partial_q^T F_1 & \vec\partial_{q_0}\vec\partial_{q_0}^T F_1\end{pmatrix}$$

$$F=\begin{pmatrix}1 & 0\\ -F_{11} & -F_{12}\end{pmatrix},\quad G=\begin{pmatrix}0 & 1\\ F_{21} & F_{22}\end{pmatrix}\;\Rightarrow\; G^{-1}=\begin{pmatrix}-F_{21}^{-1}F_{22} & F_{21}^{-1}\\ 1 & 0\end{pmatrix}$$

$$\underline{M}(\vec g)=FG^{-1}=\begin{pmatrix}-F_{21}^{-1}F_{22} & F_{21}^{-1}\\ F_{11}F_{21}^{-1}F_{22}-F_{12} & -F_{11}F_{21}^{-1}\end{pmatrix}$$

$$\underline{M}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\underline{M}^T$$

→ The map from a generating function is symplectic.

$$=\begin{pmatrix}-F_{21}^{-1} & -F_{21}^{-1}F_{22}\\ F_{11}F_{21}^{-1} & F_{11}F_{21}^{-1}F_{22}-F_{12}\end{pmatrix}\begin{pmatrix}-F_{22}F_{12}^{-1} & F_{22}F_{12}^{-1}F_{11}-F_{21}\\ F_{12}^{-1} & -F_{12}^{-1}F_{11}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

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SP(2N) ↦ F

Symplectic tranport maps have a Generating Functions

$$\vec{z} = \vec{M}(\vec{z}_0)$$

$$\begin{pmatrix} \vec{q} \\ \vec{q}_0 \end{pmatrix} = \begin{pmatrix} \vec{M}_1(\vec{z}_0) \\ \vec{q}_0 \end{pmatrix} = \vec{l}(\vec{z}_0)\,, \quad \begin{pmatrix} \vec{p}_0 \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{p}_0 \\ \vec{M}_2(\vec{z}_0) \end{pmatrix} = \vec{h}(\vec{z}_0) = \underline{J}\left[\vec{\partial} F_1(\vec{q},\vec{q}_0)\right]_{\vec{l}(\vec{z}_0)}$$

$$\vec{\partial} F_1 = -\underline{J}\vec{h} \circ \vec{l}^{\,-1} = \vec{F}$$

For F_1 to exist it is necessary and sufficient that $\partial_i F_j = \partial_j F_i \quad \Rightarrow \quad \underline{F} = \underline{F}^T$

$$-\underline{J}\vec{h} = \vec{F} \circ \vec{l} \quad \Rightarrow \quad -\underline{J}\,\underline{h} = \underline{F}(\vec{l}\,)\,\underline{l}$$

Is $\underline{J}\ \underline{h}\ \underline{l}^{-1}$ symmetric ? Yes since:

$$\underline{J}\,\underline{h}\,\underline{l}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \vec{\partial}^T_{q_0}\vec{M}_2 & \vec{\partial}^T_{p_0}\vec{M}_2 \end{pmatrix} \begin{pmatrix} \vec{\partial}^T_{q_0}\vec{M}_1 & \vec{\partial}^T_{p_0}\vec{M}_1 \\ 1 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} M_{21} & M_{22} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ M_{12}^{-1} & -M_{12}^{-1}M_{11} \end{pmatrix} = \begin{pmatrix} M_{22}M_{12}^{-1} & M_{21} - M_{22}M_{12}^{-1}M_{11} \\ M_{12}^{-1} & M_{12}^{-1}M_{11} \end{pmatrix}$$

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$$\underline{J}\,\underline{h}\,\underline{l}^{-1} = \begin{pmatrix} M_{22}M_{12}^{-1} & M_{21} - M_{22}M_{12}^{-1}M_{11} \\ M_{12}^{-1} & M_{12}^{-1}M_{11} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\vec{M}(\vec{z}_0) = \begin{pmatrix} \vec{M}_1(\vec{q}_0, \vec{p}_0) \\ \vec{M}_2(\vec{q}_0, \vec{p}_0) \end{pmatrix}, \quad \underline{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$\underline{M}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\underline{M}^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} -M_{12} & M_{11} \\ -M_{22} & M_{21} \end{pmatrix}\begin{pmatrix} M_{11}^T & M_{21}^T \\ M_{12}^T & M_{22}^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$M_{12}M_{11}^T = M_{11}M_{12}^T \qquad \Rightarrow \qquad (M_{12}^{-1}M_{11})^T = [M_{12}^{-1}M_{11}M_{12}^T]M_{12}^{-T} = M_{12}^{-1}M_{11}$$

$$M_{21}M_{22}^T = M_{22}M_{21}^T$$

$$D = D^T$$

$$M_{11}M_{22}^T - M_{12}M_{21}^T = 1$$

$$M_{22}M_{11}^T - M_{21}M_{12}^T = 1$$

$$A = A^T$$

$$(M_{22}M_{12}^{-1})^T = [M_{22}M_{11}^T M_{12}^{-T} - M_{21}]M_{22}^T = M_{22}[M_{12}^{-1}M_{11}M_{22}^T - M_{21}^T] = M_{22}M_{12}^{-1}$$

$$M_{21} - M_{22}M_{12}^{-1}M_{11} = M_{21} - M_{22}M_{11}M_{12}^{-T} = M_{12}^{-T} \longrightarrow B = C^T$$

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Symplectic Representations

Hamiltonian

$$\vec{z}' = \underline{J}\,\vec{\partial} H(\vec{z}, s)$$

Generating Functions

$$(\vec{p}, \vec{p}_0) = -\underline{J}\,\vec{\partial} F_1(\vec{q}, \vec{q}_0, s)$$

Symplectic transport map

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$$

Advantages of Symplecticity

- Determinant of the transfer matrix of linear motion is 1:

$$\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0 \quad \text{with} \quad \det(\underline{M}(s)) = +1$$

- One function suffices to compute the total nonlinear transfer map:

$$F_1(\vec{q}, \vec{q}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1(\vec{q}, \vec{q}_0, s) \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1(\vec{q}, \vec{q}_0, s)$$

$$\left.\begin{aligned} \vec{z} &= \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{q} \\ -\vec{\partial}_q F_1(\vec{q}, \vec{q}_0, s) \end{pmatrix} = \vec{f}(\vec{Q}, s) \\ \vec{z}_0 &= \begin{pmatrix} \vec{q}_0 \\ \vec{p}_0 \end{pmatrix} = \begin{pmatrix} \vec{q}_0 \\ \vec{\partial}_{q_0} F_1(\vec{q}, \vec{q}_0, s) \end{pmatrix} = \vec{g}(\vec{Q}, s) \end{aligned}\right\} \quad \begin{aligned} \vec{z} &= \vec{f}(\vec{g}^{-1}(\vec{z}_0, s), s) \\ \vec{M} &= \vec{f} \circ \vec{g}^{-1} \end{aligned}$$

- Therefore Taylor Expansion coefficients of the transport map are related.
- Computer codes can numerically approximate $\vec{M}(s, \vec{z}_0)$ with exact symplectic symmetry.
- Liouville's Theorem for phase space densities holds.

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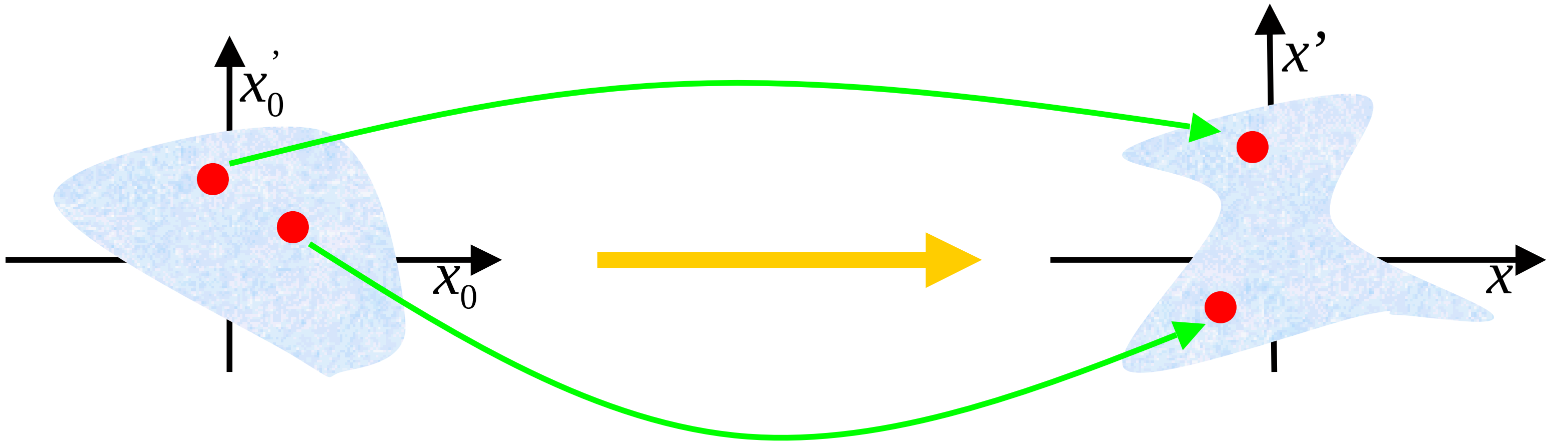
Liovile's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion. $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$ with $\det[\underline{M}(s)] = +1$



Volume = $V = \iint_V d^n\vec{z} = \iint_{V_0} \left|\frac{\partial \vec{z}}{\partial \vec{z}_0}\right| d^n\vec{z}_0 = \iint_{V_0} |\underline{M}| d^n\vec{z}_0 = \iint_{V_0} d^n\vec{z}_0 = V_0$

Hamiltonian Motion → $V = V_0$

But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$

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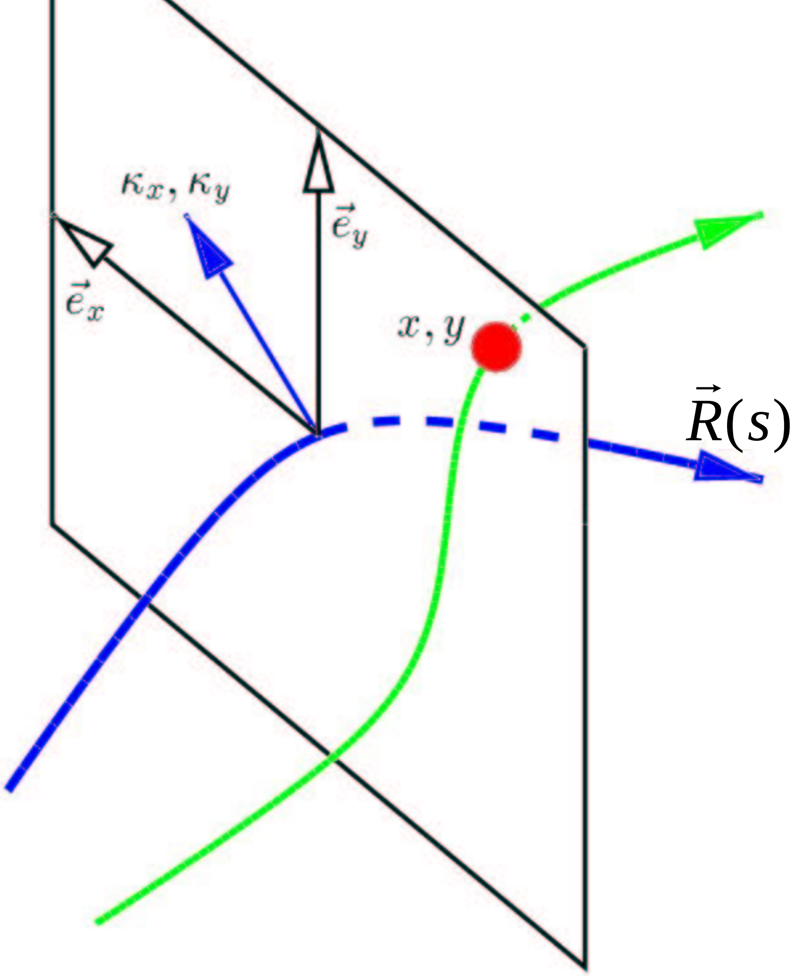
The Frenet Coordinate System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$\left|d\vec{R}\right| = ds$$

$$\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)$$

$$\vec{e}_\kappa \equiv -\tfrac{d}{ds}\vec{e}_s \Big/ \left|\tfrac{d}{ds}\vec{e}_s\right|$$

$$\vec{e}_b \equiv \vec{e}_s \times \vec{e}_\kappa$$

$$\tfrac{d}{ds}\vec{e}_s = -\kappa\,\vec{e}_\kappa \quad \text{with} \quad \kappa = \frac{1}{\rho}$$

$$0 = \tfrac{d}{ds}\left(\vec{e}_\kappa \cdot \vec{e}_s\right) = \vec{e}_s \cdot \tfrac{d}{ds}\vec{e}_\kappa - \kappa$$

Accumulated torsion angle T

$$\tfrac{d}{ds}\vec{e}_\kappa = \kappa\,\vec{e}_s + T'\vec{e}_b$$

$$0 = \tfrac{d}{ds}\left(\vec{e}_b \cdot \vec{e}_\kappa\right) = \vec{e}_\kappa \cdot \tfrac{d}{ds}\vec{e}_b + T'$$

$$\tfrac{d}{ds}\vec{e}_b = -T'\vec{e}_\kappa$$

$$\vec{r}\,' = (x' - yT')\vec{e}_\kappa + (y' + xT')\vec{e}_b + (1 + x\kappa)\vec{e}_s$$

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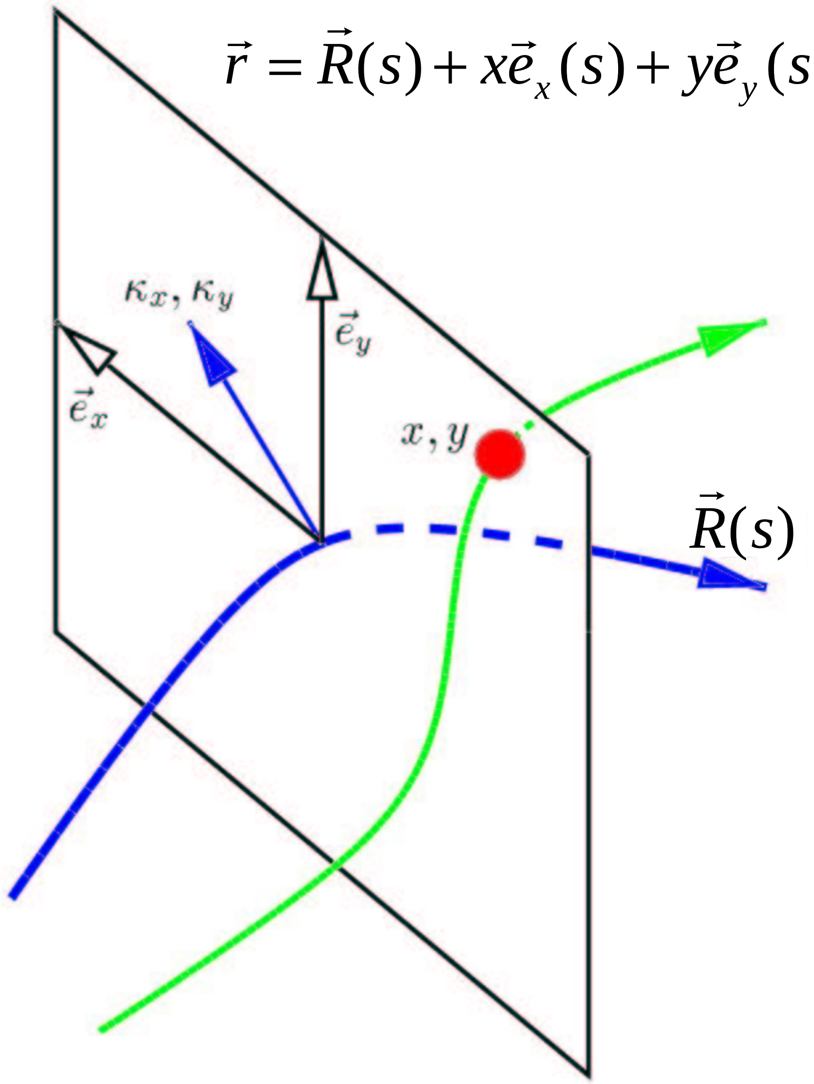
The Curvilinerar System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$\vec{e}_x \equiv \vec{e}_\kappa \cos(T) - \vec{e}_b \sin(T)$$
$$\vec{e}_y \equiv \vec{e}_\kappa \sin(T) + \vec{e}_b \cos(T)$$

$$\tfrac{d}{ds}\vec{e}_s = -\kappa_x \vec{e}_x - \kappa_y \vec{e}_y$$
$$\tfrac{d}{ds}\vec{e}_x = \kappa \cos(T)\vec{e}_s = \kappa_x \vec{e}_s$$
$$\tfrac{d}{ds}\vec{e}_y = \kappa \sin(T)\vec{e}_s = \kappa_y \vec{e}_s$$

$$\tfrac{d}{ds}\vec{r} = x'\vec{e}_\kappa + y'\vec{e}_b + (1 + x\kappa_x + y\kappa_y)\vec{e}_s$$

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Phase Space ODE

$$\frac{d}{ds}\vec{r} = x'\vec{e}_x + y'\vec{e}_y + \underbrace{(1 + x\kappa_x + y\kappa_y)}_{h}\vec{e}_s$$

$$\frac{d^2}{dt^2}\vec{r} = \vec{F}$$

$$\frac{d}{ds}\vec{r} = \dot{s}^{-1}\frac{d}{dt}\vec{r} = \dot{s}^{-1}\frac{1}{m\gamma}\vec{p} = \frac{h}{p_s}\vec{p}$$

$$\frac{d}{ds}\vec{p} = (p_x' - p_s\kappa_x)\vec{e}_x + (p_y' - p_s\kappa_y)\vec{e}_y + (p_s' + \kappa_x p_x + \kappa_y p_y)\vec{e}_s$$

$$= \dot{s}^{-1}\frac{d}{dt}\vec{p} = \dot{s}^{-1}\vec{F} = \frac{m\gamma h}{p_s}\vec{F}$$

$$\begin{pmatrix} x' \\ y' \\ p_x' \\ p_y' \end{pmatrix} = \begin{pmatrix} \frac{h}{p_s} p_x \\ \frac{h}{p_s} p_y \\ \frac{m\gamma h}{p_s} F_x + p_s\kappa_x \\ \frac{m\gamma h}{p_s} F_y + p_s\kappa_y \end{pmatrix}$$

$$t' = \dot{s}^{-1} = \frac{hm\gamma}{p_s}$$

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

$$E' = \frac{d}{dp}\sqrt{(pc)^2 + (mc^2)^2}\,\frac{d}{ds}p = c^2\frac{\vec{p}}{E}\frac{d}{ds}\vec{p} = \frac{h}{p_s}\vec{p}\cdot\vec{F}$$

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6 Dimensional Phase Space

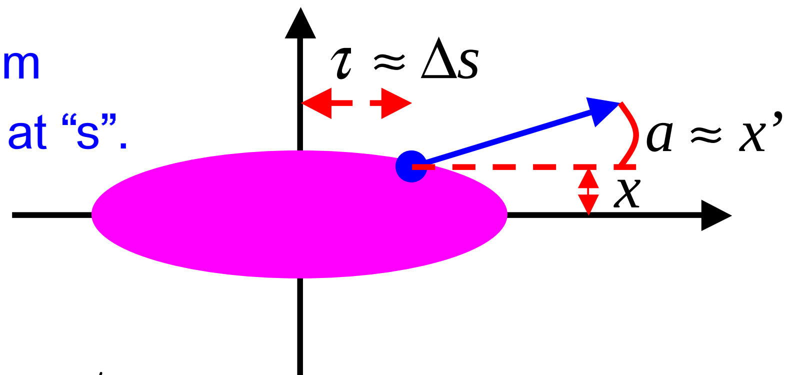
Using a reference momentum p_0 and a reference time t_0:

$$\vec{z} = (x, a, y, b, \tau, \delta)$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{c^2}{v_0} = (t_0 - t)\frac{E_0}{p_0}$$

Usually p_0 is the design momentum of the beam

And t_0 is the time at which the bunch center is at “s”.



$$\left.\begin{aligned} x' &= \partial_{p_x} K \\ p_x' &= -\partial_x K \end{aligned}\right\} \Rightarrow \begin{cases} x' = \partial_a K/p_0 , & a' = -\partial_x K/p_0 \\ y' = \partial_b K/p_0 , & b' = -\partial_y K/p_0 \end{cases}$$

$$-t' = \partial_E K \quad \Rightarrow \quad \tau' = \tfrac{c^2}{v_0}\partial_\delta K/E_0 = \partial_\delta K/p_0$$

$$E' = -\partial_{-t} K \quad \Rightarrow \quad \delta' = -\tfrac{1}{E_0}\partial_\tau K \tfrac{c^2}{v_0} \quad = -\partial_\tau K/p_0$$

New Hamiltonian:

$$\tilde{H} = K/p_0$$

The Equation of Motion

$$\begin{pmatrix} x' \\ y' \\ p_x' \\ p_y' \end{pmatrix} = \begin{pmatrix} \frac{h}{p_s} p_x \\ \frac{h}{p_s} p_y \\ \frac{m\gamma h}{p_s} F_x + p_s \kappa_x \\ \frac{m\gamma h}{p_s} F_x + p_s \kappa_x \end{pmatrix}$$

$$t' = \dot{s}^{-1} = \frac{hm\gamma}{p_s}$$

$$E' = \frac{h}{p_s} \vec{p} \cdot \vec{F}$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{E_0}{p_0}$$

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} h \frac{p_0}{p_s} a \\ \frac{m\gamma h}{p_s p_0} F_x + \frac{p_s}{p_0} \kappa_x \\ h \frac{p_0}{p_s} b \\ \frac{m\gamma h}{p_s p_0} F_y + \frac{p_s}{p_0} \kappa_x \\ \frac{E_0}{p_0}\left(\frac{m\gamma_0}{p_0} - h \frac{m\gamma}{p_s}\right) \\ \frac{h}{E_0 p_s} \vec{p} \cdot \vec{F} \end{pmatrix} = \begin{pmatrix} h \frac{p_0}{p_s} a \\ \frac{h}{p_s p_0} q(m\gamma E_x + p_y B_s - p_s B_y) + \frac{p_s}{p_0} \kappa_x \\ h \frac{p_0}{p_s} b \\ \frac{h}{p_s p_0} q(m\gamma E_y + p_s B_x - p_x B_s) + \frac{p_s}{p_0} \kappa_y \\ \frac{c^2}{v_0^2} - h \frac{c^2}{v_0 v_s} \\ \frac{h}{E_0 p_s} q(p_x E_x + p_y E_y + p_s E_s) \end{pmatrix}$$

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The 0th Order Equation of Motion

One expands around the reference trajectory:

Condition: The reference or design trajectory can be the path of a particle.
The particle transport is then origin preserving.

$$\vec{z}' = \vec{F}(\vec{z}, s) \quad \text{with} \quad \vec{F}(\vec{0}, s) = \vec{0} \quad \Rightarrow \quad \vec{M}(\vec{0}, s) = \vec{0}$$

0th order: $\vec{E} = \vec{E}_0 + \vec{E}_1 + \dots$

$$\kappa_x = \frac{q}{p_0} B_{y0} - \frac{q}{p_0 v_0} E_{x0}$$

Note: q/p_0 called magnetic rigidity
$q/(p_0 v_0)$ called electric rigidity

$$\kappa_y = -\frac{q}{p_0} B_{x0} - \frac{q}{p_0 v_0} E_{y0}$$

$E_{s0} = 0$ (No acceleration on the design trajectory)

If the energy E changes on the reference trajectory then
δ = E-E_0 does not stay 0. One then works with p_x, p_y, and E rather than with a, b, and δ.

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The Linear Equation of Motion

$$p = \frac{1}{c}\sqrt{E^2 - (mc^2)^2} \quad \Rightarrow \quad \frac{dp}{dE} = \frac{E}{pc^2} = \frac{1}{v}$$

$$p_s = \sqrt{p^2 - p_x^2 - p_y^2} = p_0[1 + \beta_0^{-2}\delta] + O^2$$

$$v_s = \tfrac{v}{p} p_s = \tfrac{c^2}{E} p_s = v_0[1 + \beta_0^{-2}\delta] + O^2 =$$

1st order: $x' = h\frac{p_0}{p_s}a \quad =_1 a\,, \quad y' = h\frac{p_0}{p_s}b \quad =_1 b$

$$\tau' = \tfrac{c^2}{v_0^2} - h\tfrac{c^2}{v_0 v_s} =_1 -\beta_0^{-2}(x\kappa_x + y\kappa_y) + \tfrac{1}{\gamma_0^2}\beta_0^{-4}\delta$$

$$a' = \tfrac{h}{p_s p_0} q(m\gamma E_x + p_y B_s - p_s B_y) + \tfrac{p_s}{p_0}\kappa_x$$

$$= -(x\kappa_x + y\kappa_y)\kappa_x + \tfrac{q}{p_0}(\tfrac{1}{v_0}E_{x1} + bB_{s0} - B_{y1}) + \delta\beta_0^{-2}[\kappa_x - \beta_0^{-2} qE_{x0}]$$

$$b' = \ldots$$

$$\delta' = \tfrac{h}{E_0 p_s} q(p_x E_x + p_y E_y + p_s E_s) = \tfrac{1}{E_0} q(aE_{x0} + bE_{y0} + E_{s1})$$

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Simplified Equation of Motion

Only bend in the horizontal plane: $\kappa_y = 0\,,\quad \kappa_x = \kappa = 1/\rho$

Only magnetic fields: $\vec{E} = 0$

Mid-plane symmetry: $B_x(x,y,s) = -B_x(x,-y,s)\,,\quad B_y(x,y,s) = B_y(x,-y,s)$

$$a' = -x\kappa^2 - \tfrac{q}{p_0}\partial_x B_y x + \delta\beta_0^{-2}\kappa \quad\Rightarrow\quad x'' = -x\,(\kappa^2 + k) + \delta\beta_0^{-2}\kappa$$

$$b' = \tfrac{q}{p_0}\partial_y B_x x \quad\Rightarrow\quad y'' = k\,y$$

$$\tau' = -x\,\beta_0^{-2}\kappa + \tfrac{1}{\gamma_0^2}\beta_0^{-4}\delta$$

$$\delta' = 0$$

Hamiltonian:

$$H = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 + \tfrac{1}{2}k(x^2 - y^2) + \tfrac{1}{2}\kappa^2 x^2 - \beta_0^{-2}\kappa\,x\delta + \tfrac{1}{2}\tfrac{1}{\gamma_0^2}\beta_0^{-4}\delta^2$$

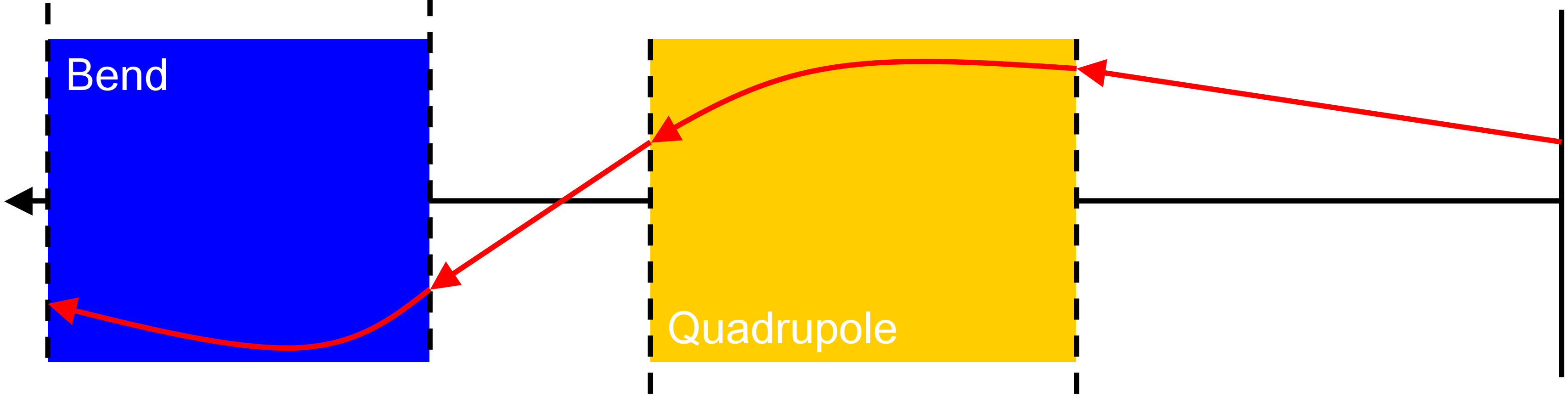
Matrix Solutions

Linear equation of motion: $\vec{z}' = \underline{F}(s)\vec{z}$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$

$$\vec{z} = \underline{M}_{\text{bend}}(L_4)\underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$



$$\vec{z} = \underline{M}_{\text{drift}}(L_1)\vec{z}_0 \qquad \vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

The Drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ \frac{1}{\gamma_0^2}\beta_0^{-4}\delta \\ 0 \end{pmatrix}$$

Note that in nonlinear expansion $x' \neq a$ so that the drift does not have a linear transport map even though $x(s) = x_0 + x_0' s$ is completely linear.

$$\frac{1}{\gamma^2} << 1 \quad \Rightarrow \quad \begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

The Dipole Equation of Motion

$$\frac{1}{\gamma^2} << 1 \quad \Rightarrow \quad \begin{aligned} x'' &= -x\,\kappa^2 + \delta\kappa \\ y'' &= 0 \\ \tau' &= -x\,\kappa \end{aligned}$$

Homogeneous solution:

$$x_H'' = -x_H\,\kappa^2 \quad \Rightarrow \quad x_H = A\cos(\kappa s) + B\sin(\kappa s)$$ (natural ring focusing)

Variation of constants:

$$x = A(s)\cos(\kappa\, s) + B(s)\sin(\kappa\, s)$$

$$x' = -A\kappa\sin(\kappa s) + B\kappa\cos(\kappa s) + \underbrace{A'\cos(\kappa s) + B'\sin(\kappa s)}_{\equiv 0}$$

$$x'' = -\kappa^2 x \underbrace{- A'\kappa\sin(\kappa s) + B'\kappa\cos(\kappa s)}_{=\delta\kappa} = -\kappa^2 x + \delta\kappa$$

$$\begin{pmatrix} \cos(\kappa s) & \sin(\kappa s) \\ -\sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 0 \\ \delta\beta_0^{-2} \end{pmatrix}$$

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The Dipole

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$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \cos(\kappa s) & -\sin(\kappa s) \\ \sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \delta\,\kappa^{-1} \begin{pmatrix} \cos(\kappa s) \\ \sin(\kappa s) \end{pmatrix} + \begin{pmatrix} A_H \\ B_H \end{pmatrix} \quad \text{with} \quad x = A\cos(\kappa s) + B\sin(\kappa s)$$

$$\tau' = -x\kappa$$

$$\underline{M} = \begin{pmatrix} \begin{matrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{matrix} & \underline{0} & \begin{matrix} 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ 0 & \sin(\kappa s) \end{matrix} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \begin{matrix} -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] \\ 0 & 0 \end{matrix} & \underline{0} & \begin{matrix} 1 & \kappa^{-1}[\sin(\kappa s)-s\kappa] \\ 0 & 1 \end{matrix} \end{pmatrix}$$

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Time of Flight from Symplecticity

$$\underline{M} = \begin{pmatrix} \underline{M}_4 & \vec{0} & \vec{D} \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix}$$

is in SU(6) and therefore $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 & -\vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 & -M_{56} & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix}\begin{pmatrix} \underline{M}_4^T & \vec{T} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{D}^T & M_{56} & 1 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 \underline{M}_4^T & \underline{M}_4 \underline{J}_4 \vec{T} - \vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 \underline{M}_4^T + \vec{D}^T & 0 & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\vec{T} = -\underline{J}_4 \underline{M}_4^{-1} \vec{D}$$

It is sufficient to compute the 4D map $\underline{M}_4$, the Dispersion $\vec{D}$ and the time of flight term M_{56}

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The Quadrupole

$$x'' = -x\,k$$

$$y'' = \;\; y\,k$$

$$\underline{M}_4 = \begin{pmatrix} \cos(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}\,s) & & \\ -\sqrt{k}\sin(\sqrt{k}\,s) & \cos(\sqrt{k}\,s) & & \underline{0} \\ & & \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ & \underline{0} & \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

As for a drift:

$$\vec{D} = \vec{0} \quad \Rightarrow \quad \vec{T} = \vec{0}$$

$$M_{56} = 0$$

For k<0 one has to take into account that

$$\cos(\sqrt{k}\,s) = \cosh(\sqrt{|k|}\,s)\,, \quad \sin(\sqrt{k}\,s) = i\sinh(\sqrt{|k|}\,s)$$

$$\cosh(\sqrt{k}\,s) = \cos(\sqrt{|k|}\,s)\,, \quad \sinh(\sqrt{k}\,s) = i\sin(\sqrt{|k|}\,s)$$

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The Combined Function Bend

$$x'' = -x\,(\underbrace{\kappa^2 + k}_{K}) + \delta\kappa$$

$$y'' = y\,k \quad , \quad \tau' = -\kappa\, x$$

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{0} & \underline{0} & \underline{M}_\tau \end{pmatrix}$$

$$\underline{M}_\tau = \begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix}$$

$$M_{56} = \tfrac{\kappa^2}{K\sqrt{K}}[\sin(\sqrt{K}s) - \sqrt{K}s]$$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \tfrac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \tfrac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \tfrac{\kappa}{K}[1 - \cos(\sqrt{K}s)] \\ \tfrac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix}$$

Options:

- For k>0:
  focusing in x, defocusing in y.
- For k<0, K<0:
  defocusing in x, focusing in y.
- For k<0, K>0:
  weak focusing in both planes.

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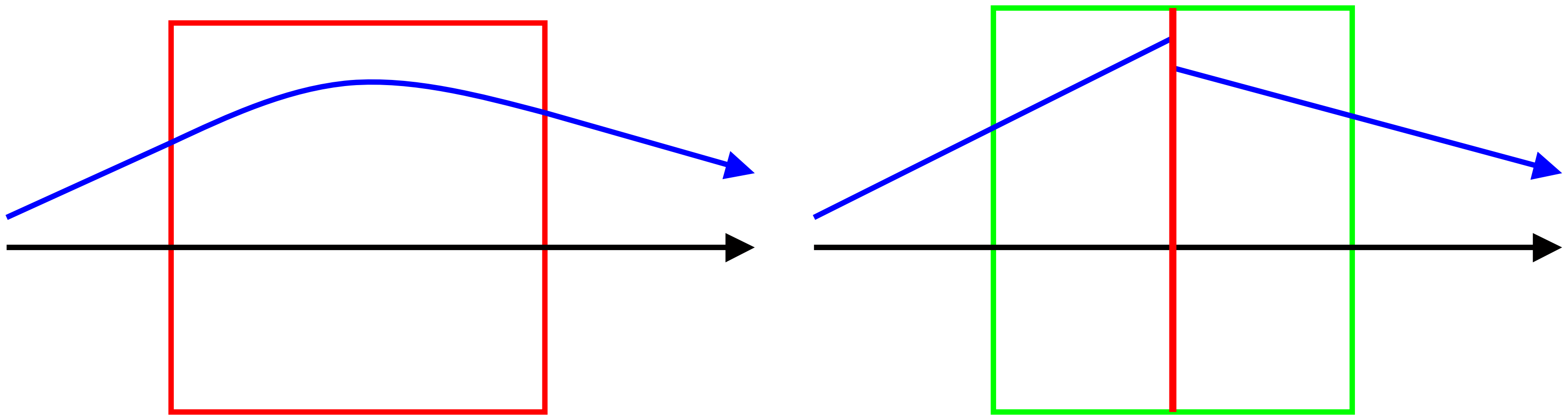
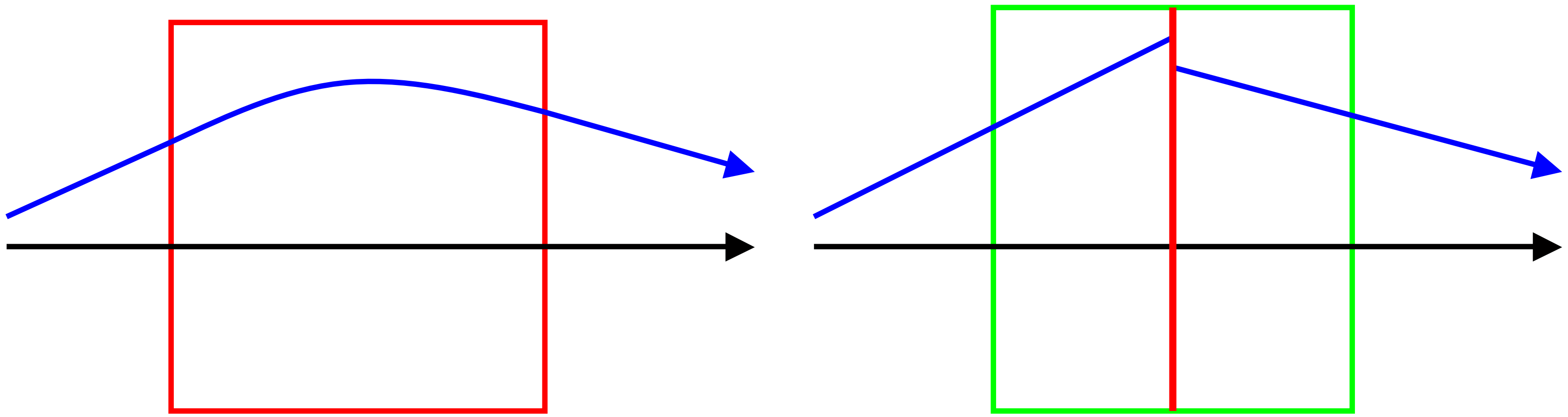
The Thin Lens Approximation



$$\vec{z}(s) = \underline{M}(s)\,\vec{z}_0 = \underline{D}(\tfrac{s}{2})\underline{D}^{-1}(\tfrac{s}{2})\underline{M}(s)\underline{D}^{-1}(\tfrac{s}{2})\underline{D}(\tfrac{s}{2})\vec{z}_0$$

Drift: $$\underline{M}^{\text{thin}}_{\text{drift}}(s) = \underline{D}^{-1}(\tfrac{s}{2})\underline{D}(s)\underline{D}^{-1}(\tfrac{s}{2}) = \underline{1}$$

The Thin Lens Quadrupole



$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) = \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{k}s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\sqrt{k}s) + \frac{s}{2}\sqrt{k}\sin(\sqrt{k}s) & (1-k\frac{s^2}{4})\frac{1}{\sqrt{k}}\sin(\sqrt{k}s) - s\cos(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) + \sqrt{k}\,\frac{s}{2}\sin(\sqrt{k}s) \end{pmatrix}$$

Weak magnet limit: $\sqrt{k}s << 1$

$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) \approx \begin{pmatrix} 1 & 0 \\ -ks & 1 \end{pmatrix}$$

The Thin Lens Dipole

$$\underline{M} = \begin{pmatrix} \begin{matrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{matrix} & \underline{0} & \begin{matrix} 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ 0 & \sin(\kappa s) \end{matrix} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \begin{matrix} -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] \\ 0 & 0 \end{matrix} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

$$\underline{M}^{\text{thin}}_{\text{bend},x\tau}(s) = \underline{D}(-\tfrac{s}{2})\underline{M}_{\text{bend},x\tau}\underline{D}(-\tfrac{s}{2}) \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\kappa^2 s & 1 & 0 & \kappa s \\ -\kappa s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Thin Combined Function Bend

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{0} & \underline{0} & \underline{1} \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix}$$

$$\underline{M}_x^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ -K\,s & 1 \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix} \longrightarrow \underline{M}_y^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ k\,s & 1 \end{pmatrix}$$

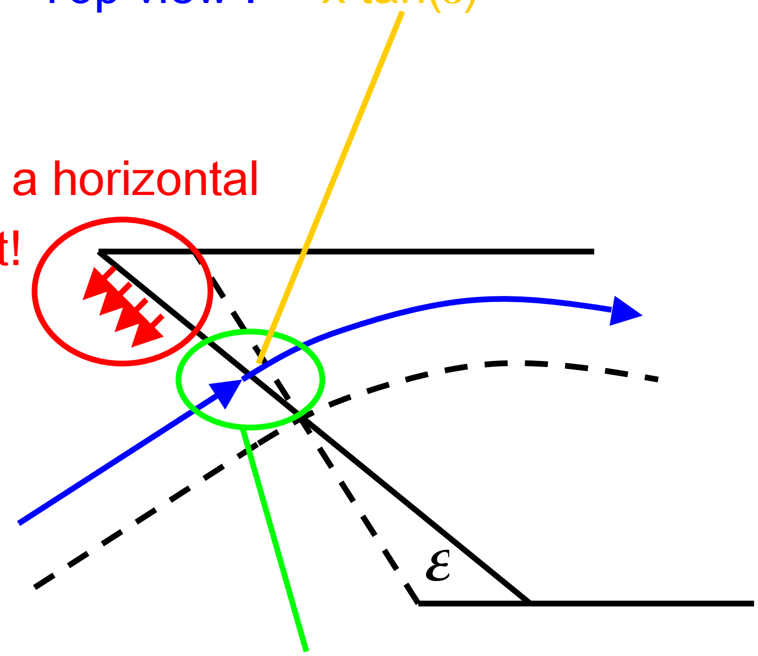
$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} 0 \\ \kappa\,s \end{pmatrix}$$

Edge Focusing

Top view :

Fringe field has a horizontal field component!



Extra bending focuses!

Horizontal focusing with $\Delta x' = -x\dfrac{\tan(\varepsilon)}{\rho}$

$$B_x = \partial_y B_s\big|_{y=0}\, y\tan(\varepsilon) = \partial_s B_y\big|_{y=0}\, y\tan(\varepsilon)$$

$$y'' = \tfrac{q}{p}\,\partial_s B_y\big|_{y=0}\, y\tan(\varepsilon)$$

$$\Delta y' = \int y'' ds = \tfrac{q}{p} B_y\, y\tan(\varepsilon) = y\frac{\tan(\varepsilon)}{\rho}$$

$$\vec{z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\tan(\varepsilon)}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\tan(\varepsilon)}{\rho} & 1 \end{pmatrix} \vec{z}_0$$

Quadrupole effect with

$$kl = \frac{\tan(\varepsilon)}{\rho}$$

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The Rectangular Bend

Together, the defocusing in the edge and the natural circle focusing compensate in the Horizontal and focus in the vertical.

$$\underline{M}^{\text{thin}}_{\text{rbend,x}} \approx \begin{pmatrix} 1 & 0 \\ \frac{1}{2}\kappa^2 s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\kappa^2 s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2}\kappa^2 s & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{M}^{\text{thin}}_{\text{rbend,y}} \approx \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\kappa^2 s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\kappa^2 s & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\kappa^2 s & 1 \end{pmatrix}$$

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Weak Focusing with Edges

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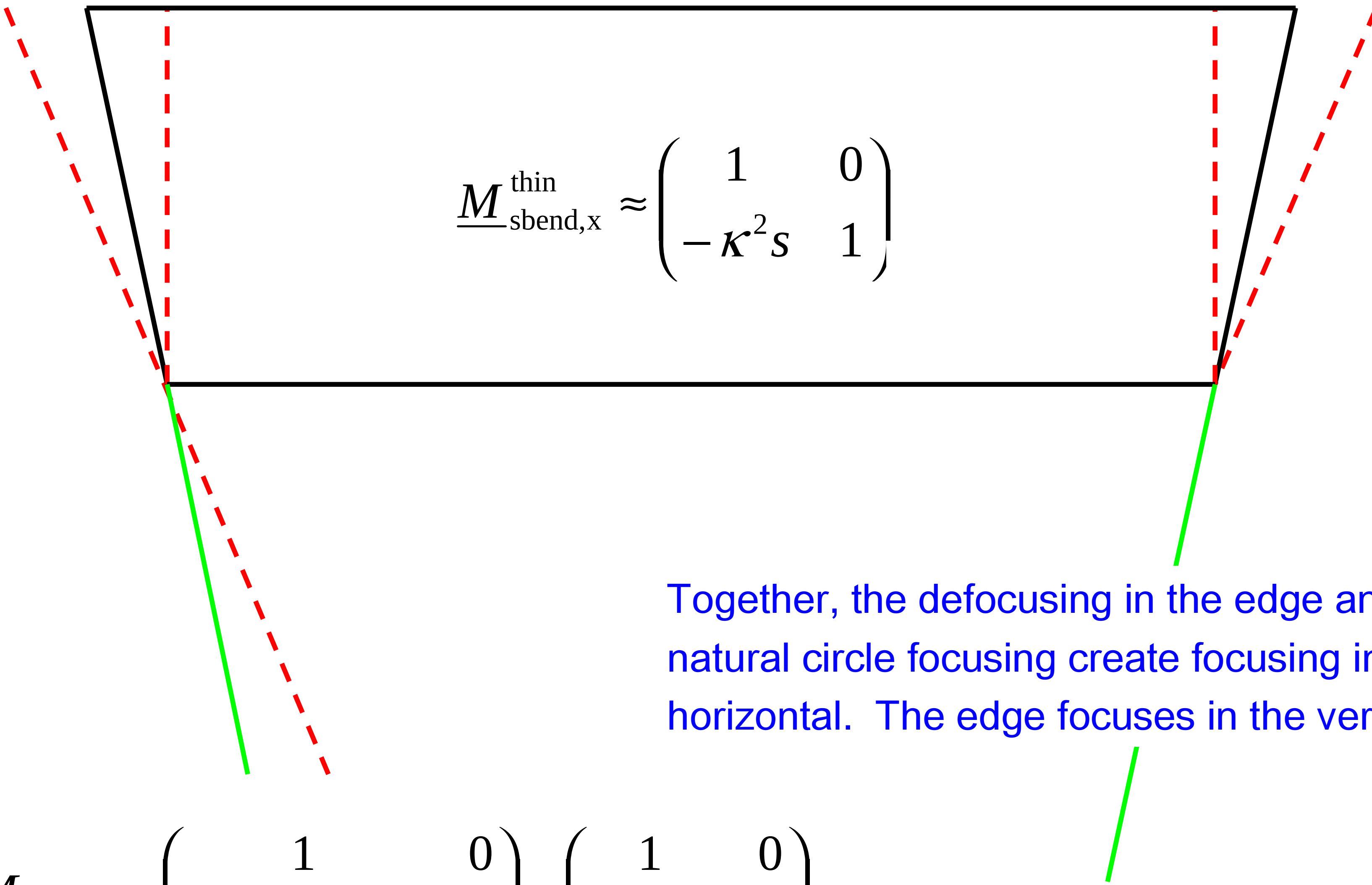

$$\underline{M}_{\text{sbend,x}}^{\text{thin}} \approx \begin{pmatrix} 1 & 0 \\ -\kappa^2 s & 1 \end{pmatrix}$$

Together, the defocusing in the edge and the natural circle focusing create focusing in the horizontal. The edge focuses in the vertical.

$$\underline{M}_{\text{edge,x}} = \begin{pmatrix} 1 & 0 \\ -\kappa \tan(\kappa \frac{s}{4}) & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ -\kappa^2 \frac{s}{4} & 1 \end{pmatrix}$$

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Cyclotrons with edge focusing

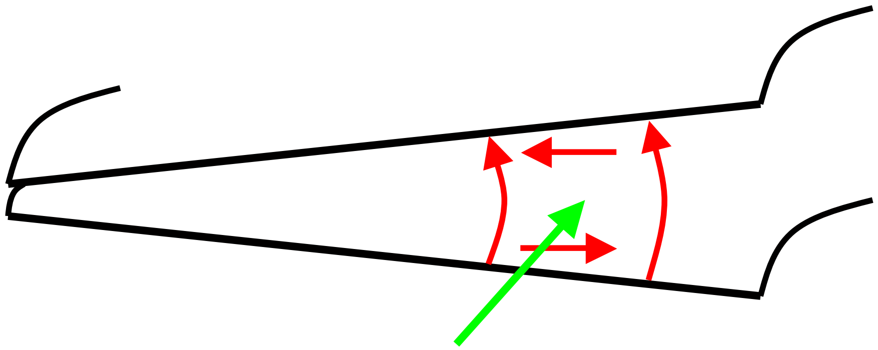
- The isocyclotron with constant

$$\omega_z = \frac{q}{m_0 \gamma(E)} B_z(r(E))$$

Up to 600MeV but

this vertically defocuses the beam.

Edge focusing is therefore used.



Variation of Constants

$$\vec{z}\,' = \vec{f}(\vec{z}, s)$$

$$\vec{z}\,' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z}, s)$

$$\vec{z}_H\,' = \underline{L}(s)\vec{z}_H \quad \Rightarrow \quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}\,'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad \Rightarrow \quad \vec{z}\,'(s) = \underline{M}\,'(s)\vec{a} + \underline{M}(s)\vec{a}\,'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}\right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s - \hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}$$

Perturbations are propagated from s to s'

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Aberrations

The deviations from the linear motion due to nonlinear forces are called Aberrations.

$$\vec{z}_1(s) = \vec{z}_H(s)$$

$$\vec{z}_2(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s-\hat{s})\Delta\vec{f}(\vec{z}_1(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}_3(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s-\hat{s})\Delta\vec{f}(\vec{z}_2(\hat{s}),\hat{s})\,d\hat{s}$$

$$w(s) = w_H(s) + \int_0^s \sum_{klmn} W_{klmn}(s,\hat{s}) w^k \overline{w}^l w'^m \overline{w}'^n d\hat{s}$$

Solenoid: C_0 symmetry $w_2(s) = w_H(s) + A(s) w_0{}^2 \overline{w}_0 + \ldots$ $\left(e^{i\phi} w_0 \Rightarrow e^{i\phi} w\right)$

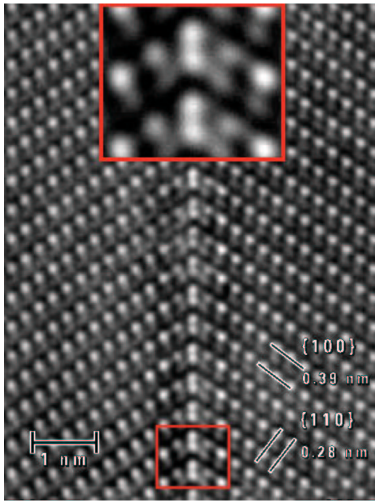
Sextupole: C_2 symmetry $w(s) = w_H(s) + \int_0^s W_{0200}(s,\hat{s}) \overline{w}^2 d\hat{s} \ldots$ $\left(e^{i\frac{2\pi}{3}} w_0 \Rightarrow e^{i\frac{2\pi}{3}} w\right)$

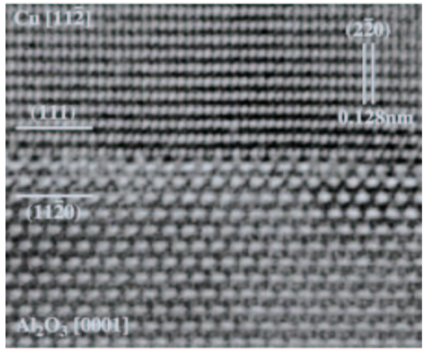
$$w_2(s) = w_H(s) + A(s)\overline{w}_0^2 + \ldots$$

$$w_2(s) = w_H(s) + A(s)\overline{w}_0^2 + B(s) w_0^2 \overline{w}_0 + \ldots$$

In second iteration, sextupoles can correct solenoid aberrations

Aberration Correction





$$w_2(s) = w_H(s) + C(s) w_0^2 \overline{w}_0 + \dots$$

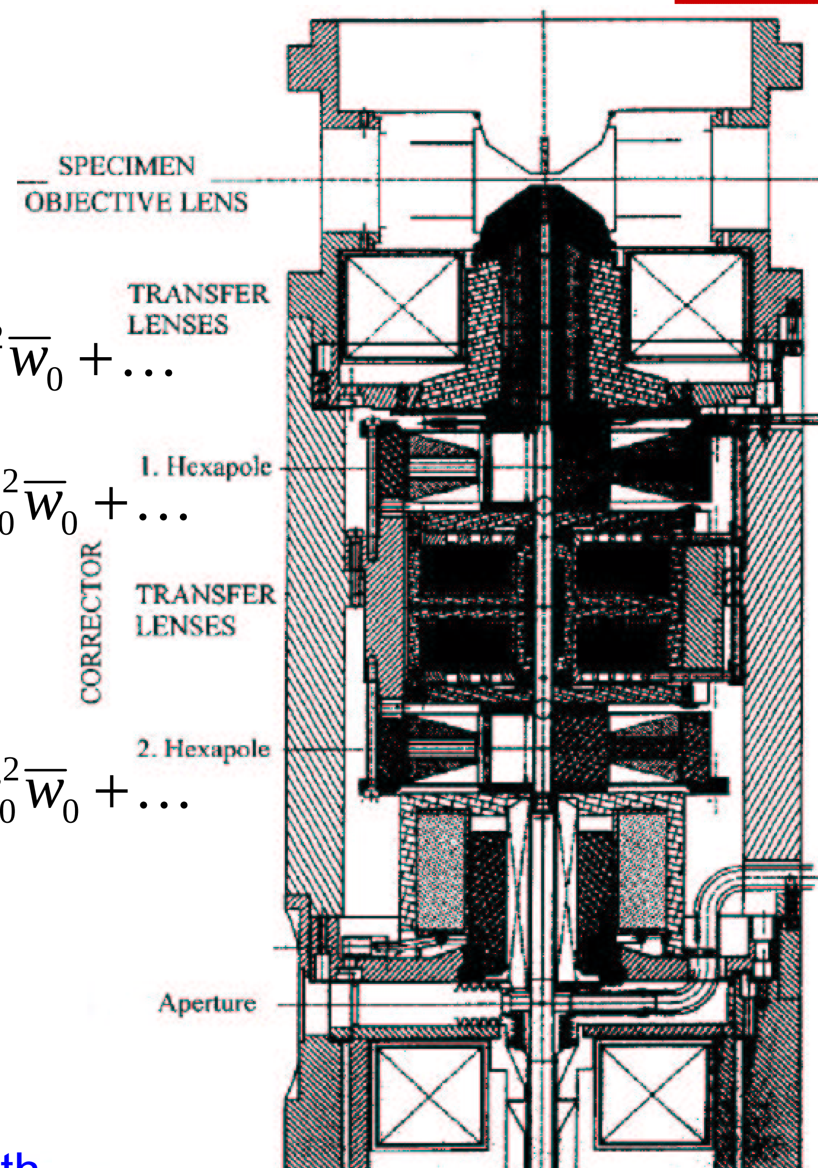
$$w_2(s) = w_H(s) + A(s) \overline{w}_0^2 + B(s) w_0^2 \overline{w}_0 + \dots$$

$$w_2(s) = w_H(s) + \cancel{A(s) \overline{w}_0^2} + 2B(s) w_0^2 \overline{w}_0 + \dots$$

2B cancels C !

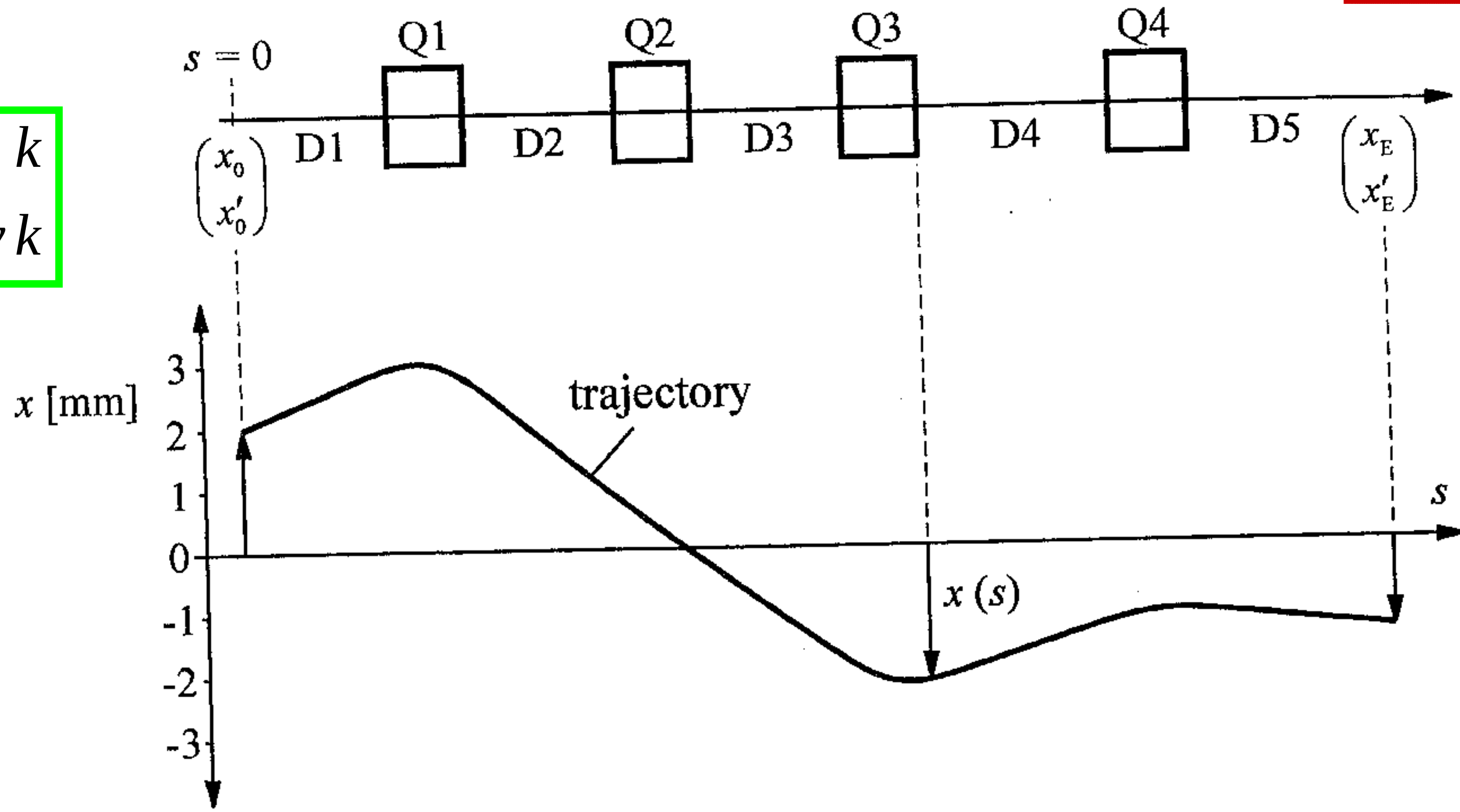
Quadratic in sextupole strength

Linear in solenoid strength



Beta Function and Betatron Phase

$$x'' = -x\,k$$
$$y'' = \;\; y\,k$$



$$x(s) = M_{11}(s)x_0 + M_{12}(s)x_0'$$
$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

Twiss Parameters

$$x'' = -k\,x$$

$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

$$x'(s) = \sqrt{\tfrac{2J}{\beta}}[\beta\psi'\cos(\psi(s)+\phi_0) - \alpha\sin(\psi(s)+\phi_0)] \quad \text{with} \quad \alpha = -\tfrac{1}{2}\beta'$$

$$x''(s) = \sqrt{\tfrac{2J}{\beta}}[(\beta\psi'' - 2\alpha\psi')\cos(\psi(s)+\phi_0) - (\alpha' + \tfrac{\alpha^2}{\beta} + \beta\psi'^2)\sin(\psi(s)+\phi_0)]$$

$$= \sqrt{\tfrac{2J}{\beta}}[-k\beta\sin(\psi(s)+\phi_0)]$$

$$\beta\psi'' - 2\alpha\psi' = \beta\psi'' + \beta'\psi' = (\beta\psi')' = 0 \quad \Rightarrow \quad \psi' = \tfrac{1}{\beta}$$

$$\alpha' + \gamma = k\beta \quad \text{with} \quad \gamma = \tfrac{1+\alpha^2}{\beta}$$

$$\beta' = -2\alpha$$

$$\alpha' = k\beta - \gamma$$

$$\psi = \int_0^s \frac{1}{\beta(s')}\,ds'$$

$\alpha, \beta, \gamma, \psi$ are called Twiss parameters.

What are the initial conditions?

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Phase Space Ellipse

Particles with a common J and different ϕ all lie on an ellipse in phase space:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

(Linear transform of a circle)

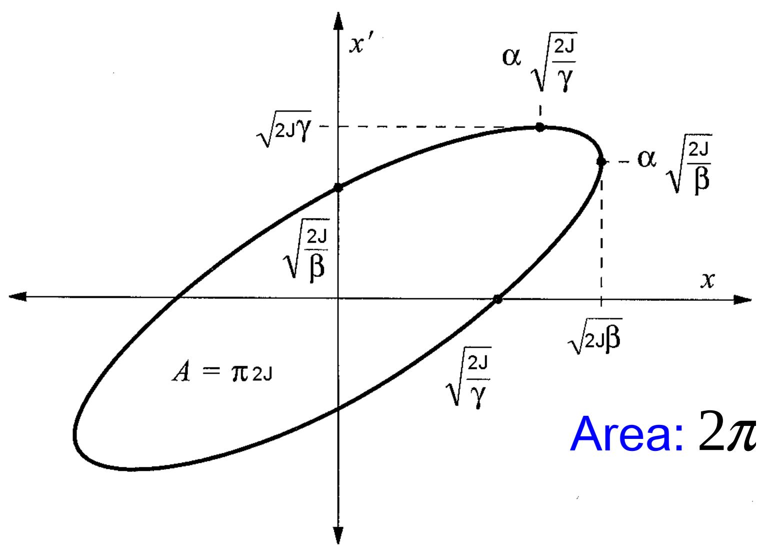
$$x_{\max} = \sqrt{2J\beta} \text{ at } x' = -\alpha\sqrt{\tfrac{2J}{\beta}}$$

$$(x, x')\begin{pmatrix} \frac{1}{\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = (x, x')\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = 2J$$

(Quadratic form)

$$\beta\gamma - \alpha^2 = 1$$

Area: $2\pi J$



What β is for x, γ is for x'

$$x'_{\max} = \sqrt{2J\gamma} \text{ at } x = -\alpha\sqrt{\tfrac{2J}{\gamma}}$$

Area: $2\pi J$ → $\int_0^{2\pi}\int_0^{J} dJ\,d\phi = 2\pi J = \iint dx\,dx'$

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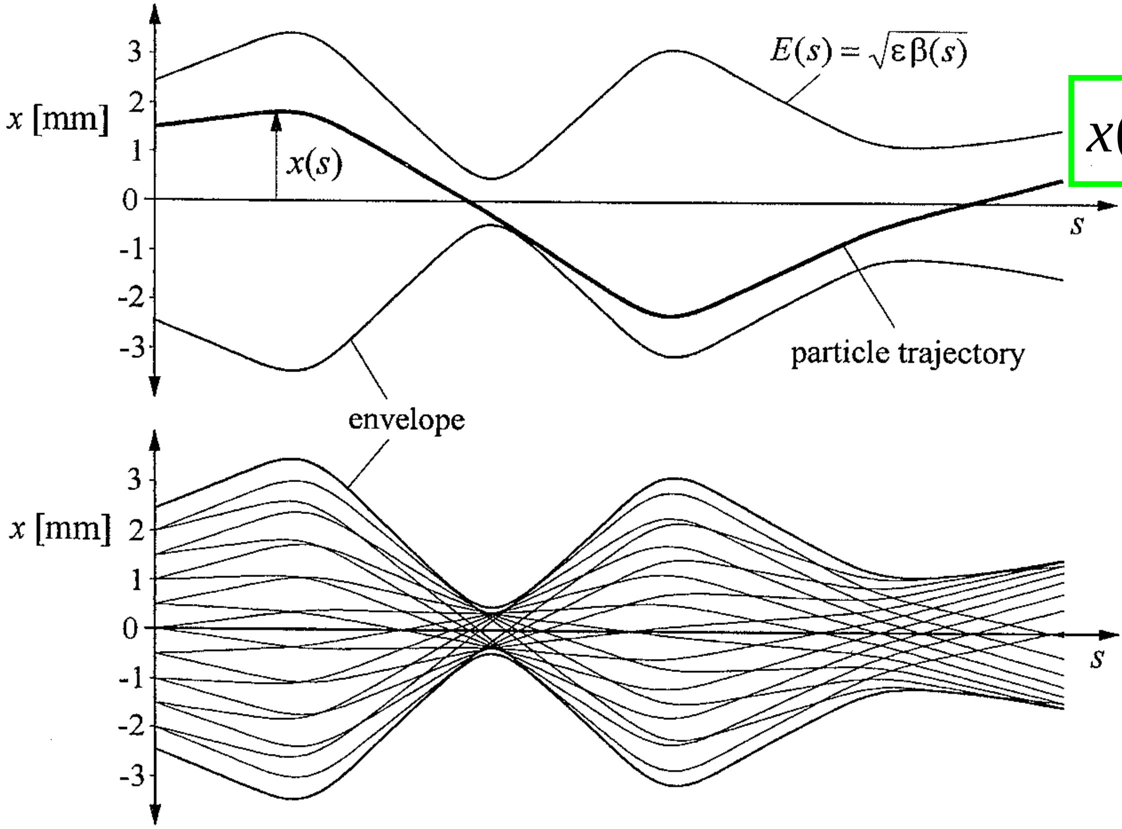
The Beam Envelope



$$x(s) = \sqrt{2J\beta(s)}\sin(\psi(s) + \phi_0)$$

In any beam there is a distribution of initial parameters. If the particles with the largest J are distributed in ϕ over all angles, then the envelope of the beam is described by $\sqrt{2J_{\max}\beta(s)}$

The initial conditions of β and α are chosen so that this is approximately the case.

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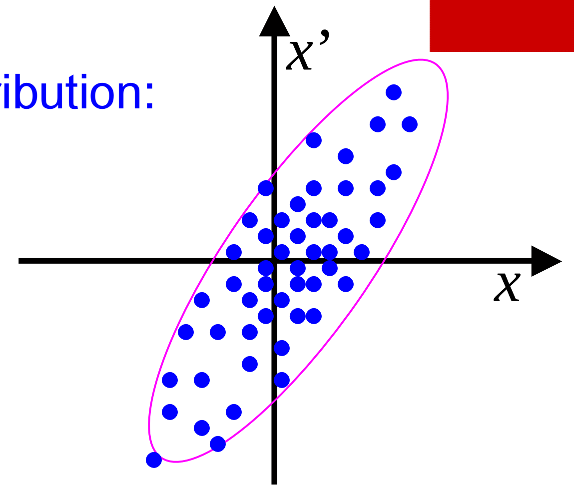
Phase Space Distribution

Often one can fit a Gauss distribution to the particle distribution:

$$\rho(x,x') = \frac{1}{2\pi\varepsilon} e^{-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\varepsilon}}$$



The equi-density lines are then ellipses. And one chooses the starting conditions for β and α according to these ellipses!

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \qquad \rho(J,\phi_0) = \frac{1}{2\pi\varepsilon} e^{-\frac{J}{\varepsilon}}$$

$$\langle 1 \rangle = \frac{1}{2\pi\varepsilon} \int_0^{2\pi}\int_0^{\infty} e^{-J/\varepsilon} dJ d\phi_0 = 1$$

Initial beam distribution ⟶ initial α, β, γ

$$\langle x^2 \rangle = \frac{1}{2\pi\varepsilon} \iint 2J\beta \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\beta \quad \longrightarrow \quad \langle x'^2 \rangle = \varepsilon\gamma$$

$$\langle x x' \rangle = -\frac{1}{2\pi\varepsilon} \iint 2J\alpha \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\alpha$$

$$\varepsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$ is called the emittance.

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Invariant of Motion

$$x(s) = \sqrt{2J\beta(s)}\sin(\psi(s) + \phi_0)$$

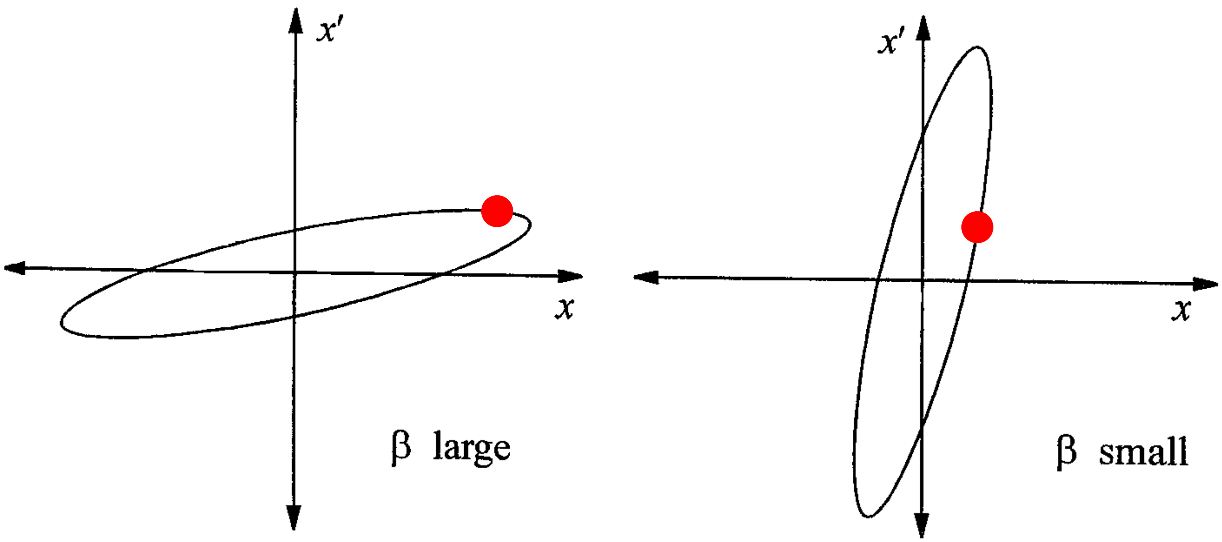
Where J and ϕ are given by the starting conditions x_0 and x'_0.

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J$$

Leads to the invariant of motion:

$$f(x, x', s) = \gamma(s) x^2 + 2\alpha(s) x x' + \beta(s) x'^2 \quad \Rightarrow \quad \tfrac{d}{ds} f = 0$$

It is called the Courant-Snyder invariant.



Propagation of Twiss Parameters

$$(x_0, x_0')\begin{pmatrix}\gamma_0 & \alpha_0\\ \alpha_0 & \beta_0\end{pmatrix}\begin{pmatrix}x_0\\ x_0'\end{pmatrix} = 2J$$

$$(x, x')\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix}\begin{pmatrix}x\\ x'\end{pmatrix} = 2J = (x_0, x_0')\underline{M}^T\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix}\underline{M}\begin{pmatrix}x_0\\ x_0'\end{pmatrix}$$

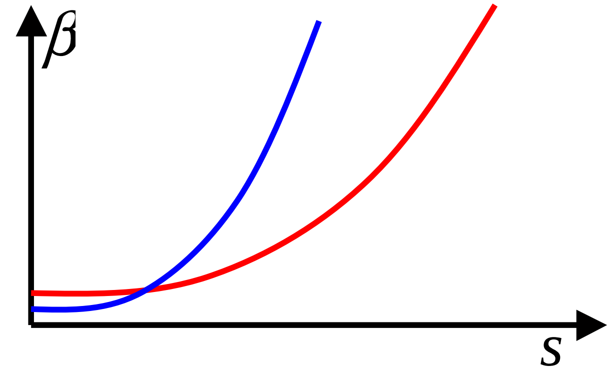
$$\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix} = \underline{M}^{-T}\begin{pmatrix}\gamma_0 & \alpha_0\\ \alpha_0 & \beta_0\end{pmatrix}\underline{M}^{-1}$$

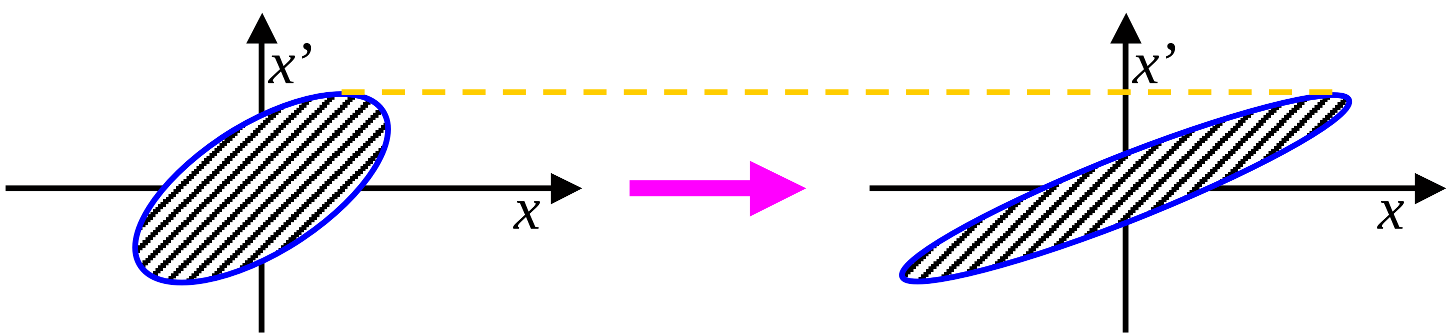
$$\begin{pmatrix}\beta & -\alpha\\ -\alpha & \gamma\end{pmatrix} = \underline{M}\begin{pmatrix}\beta_0 & -\alpha_0\\ -\alpha_0 & \gamma_0\end{pmatrix}\underline{M}^T$$

Twiss Parameters in a Drift

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 - 2\alpha_0 s + \gamma_0 s^2 & \gamma_0 s - \alpha_0 \\ \gamma_0 s - \alpha_0 & \gamma_0 \end{pmatrix}$$

$$\beta = \beta_0^* [1 + (\tfrac{s}{\beta_0^*})^2] \quad \text{for} \quad \alpha_0^* = 0$$





Twiss Parameters in a Quadrupole

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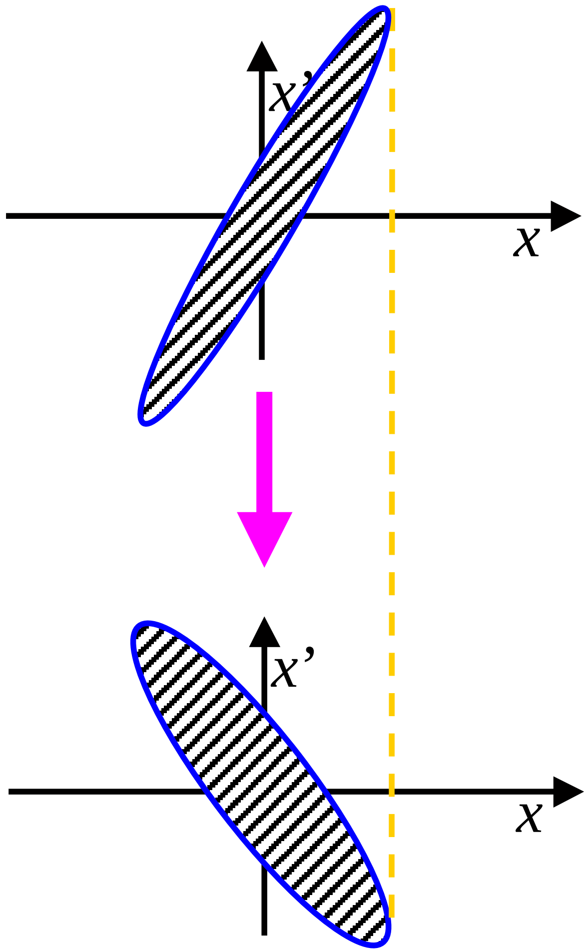
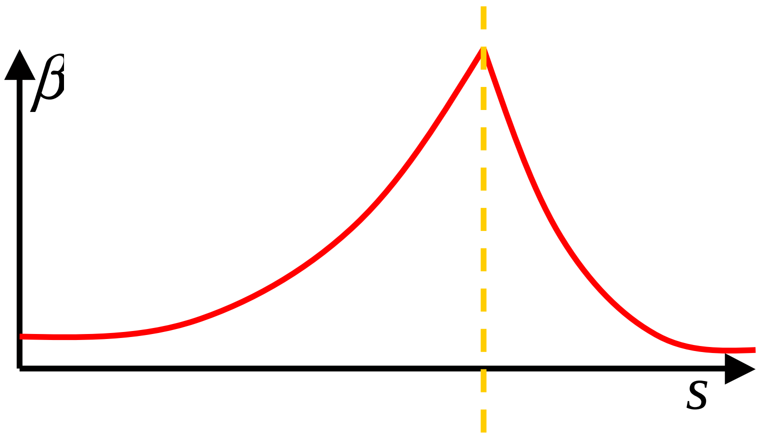


$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

$$\alpha = \alpha_0 + k\beta_0$$



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From Twiss to Transport Matrix

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta_0} & 0 \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix}\begin{pmatrix} \sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

$$= \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix}\begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix}$$

$$\underline{M}(s) = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\psi + \alpha_0 \sin\psi] & \sqrt{\beta_0\beta}\sin\psi \\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0 - \alpha)\cos\psi - (1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi - \alpha\sin\psi] \end{pmatrix}$$

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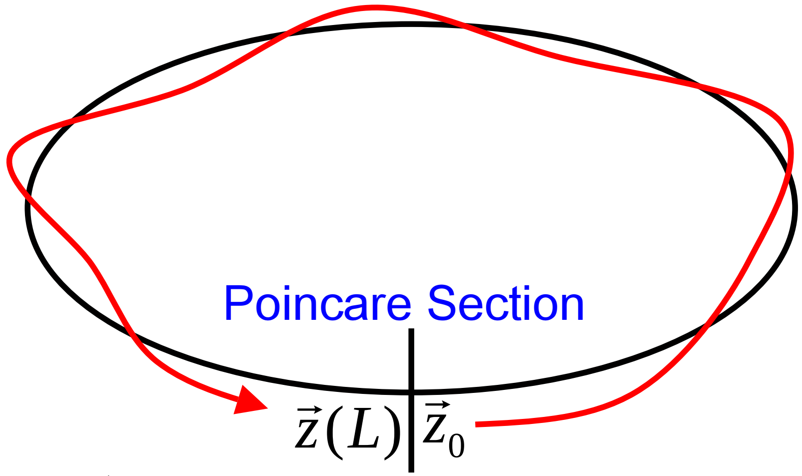
The One Turn Matrix for a Ring



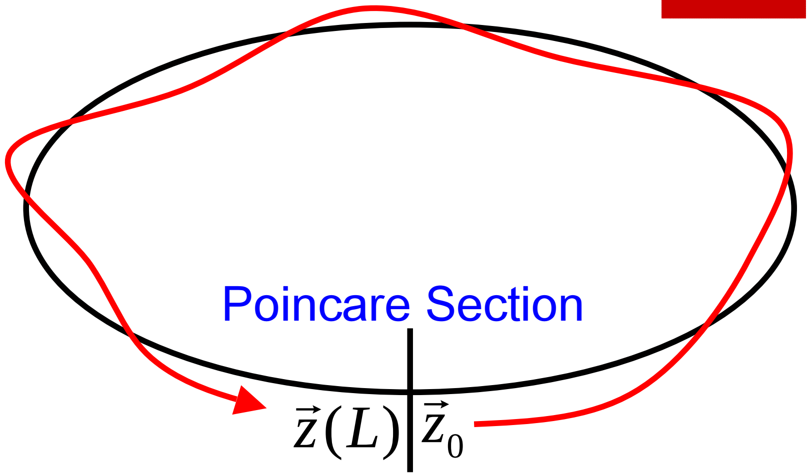
$$\vec{z}(s) = \underline{M}(s,0)\vec{z}(0)$$

$$\vec{z}(L) = \underline{M}(L,0)\vec{z}(0)$$

$$\vec{z}(s+L) = \underline{M}_0(s)\vec{z}(s) \quad , \quad \underline{M}_0 = \underline{M}(s+L,s)$$

$$\vec{z}(s+nL) = \underline{M}_0^n(s)\vec{z}(s)$$

The Periodic Beta Function



If the particle distribution in a ring is stable, it is periodic from turn to turn.

$$\rho(x,x',s+L)=\rho(x,x',s)$$

To be matched to such a beam, the Twiss parameters α, β, γ must be the same after every turn.

$$\underline{M}(s,0)=\begin{pmatrix}\sqrt{\frac{\beta}{\beta_0}}[\cos\psi+\alpha_0\sin\psi] & \sqrt{\beta_0\beta}\sin\psi\\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0-\alpha)\cos\psi-(1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi-\alpha\sin\psi]\end{pmatrix}$$

$$\underline{M}_0(s)=\begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu\\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu\end{pmatrix}=\cos\mu+\begin{pmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{pmatrix}\sin\mu$$

$$\mu=\psi(s+L)-\psi(s)$$

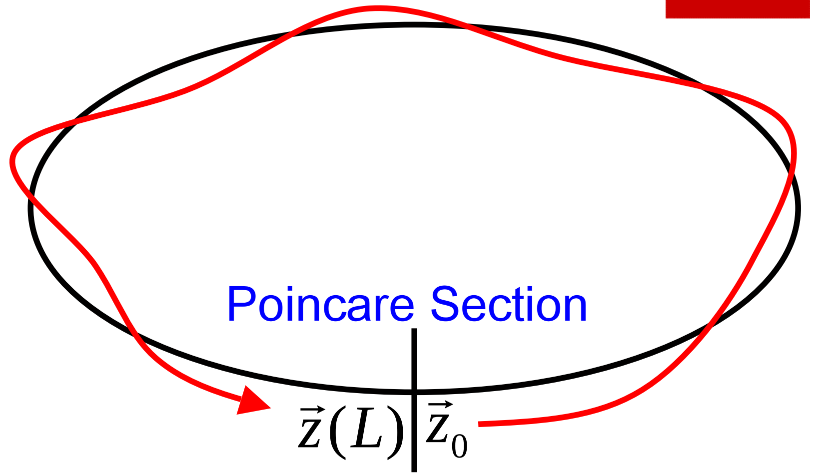
One Turn Matrix to Periodic Twiss

The periodic Twiss parameters are the solution of a nonlinear differential equation with periodic boundary conditions:

$$\beta' = -2\alpha \quad \text{with} \quad \beta(L) = \beta(0)$$

$$\alpha' = k\beta - \frac{1+\alpha^2}{\beta} \quad \text{with} \quad \alpha(L) = \alpha(0)$$

$$\mu = \int_0^L \frac{1}{\beta(\hat{s})} d\hat{s}$$



Note: $\beta(s) > 0$

$$\underline{M}_0(s) = \cos\mu + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin\mu$$

$$\cos\mu = \tfrac{1}{2}\mathrm{Tr}[\underline{M}_0(s)]$$

$$\beta = \underline{M}_{0,12} \frac{1}{\sin\mu}$$

$$\alpha = (\underline{M}_{0,11} - \underline{M}_{0,11}) \frac{1}{2\sin\mu}$$

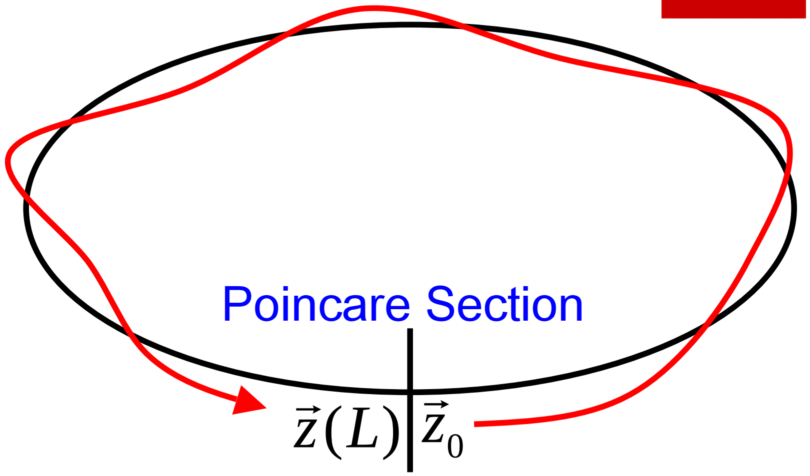
$$\gamma = \frac{1+\alpha^2}{\beta}$$

Stable beam motion and thus a periodic beta function can only exist when Tr[M]<2.

The Tune

The betatron phase advance per turn devided by 2π is called the TUNE.

$$\mu = 2\pi\nu = \psi(s+L) - \psi(s)$$

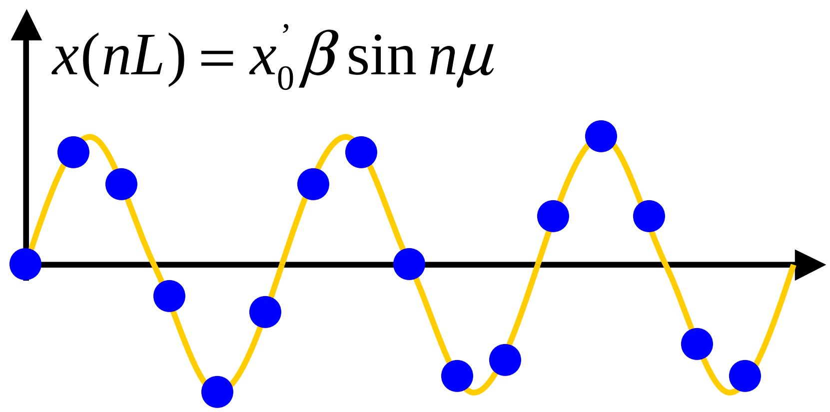


It is a property of the ring and does not depend on the azimuth s.

$$\underline{M}_0(s) = \cos\mu + \begin{pmatrix} -\alpha(s) & \beta(s) \\ \gamma(s) & \alpha(s) \end{pmatrix} \sin\mu$$

$$2\cos\mu(s) = \mathrm{Tr}[\underline{M}_0(s)] = \mathrm{Tr}[\underline{M}(s,0)\underline{M}_0(0)\underline{M}_0^{-1}(s,0)]$$

$$= \mathrm{Tr}[\underline{M}_0(0)] = 2\cos\mu(0)$$



$$\underline{M}_0^n = \cos n\mu + \begin{pmatrix} -\alpha & \beta \\ \gamma & \alpha \end{pmatrix} \sin n\mu$$

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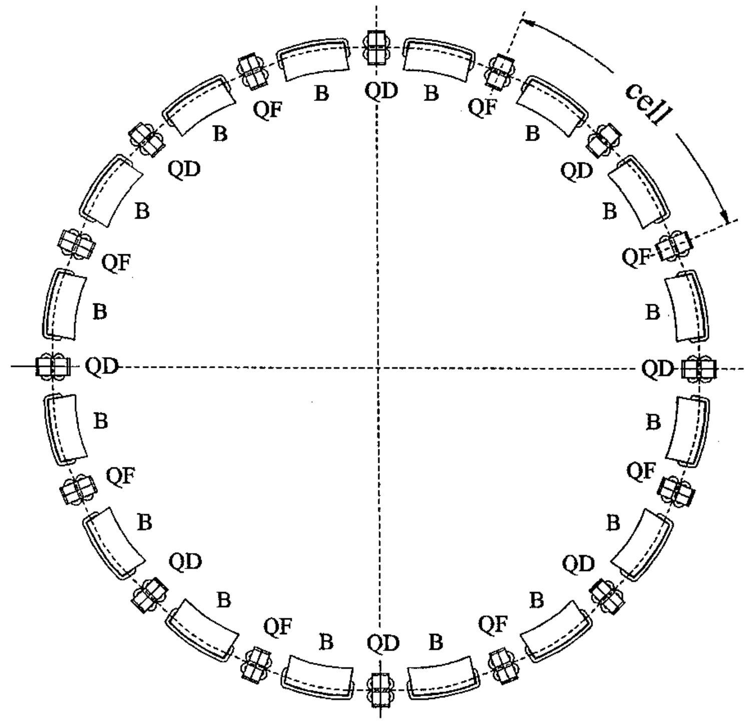
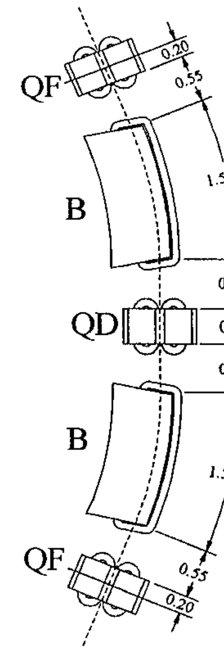
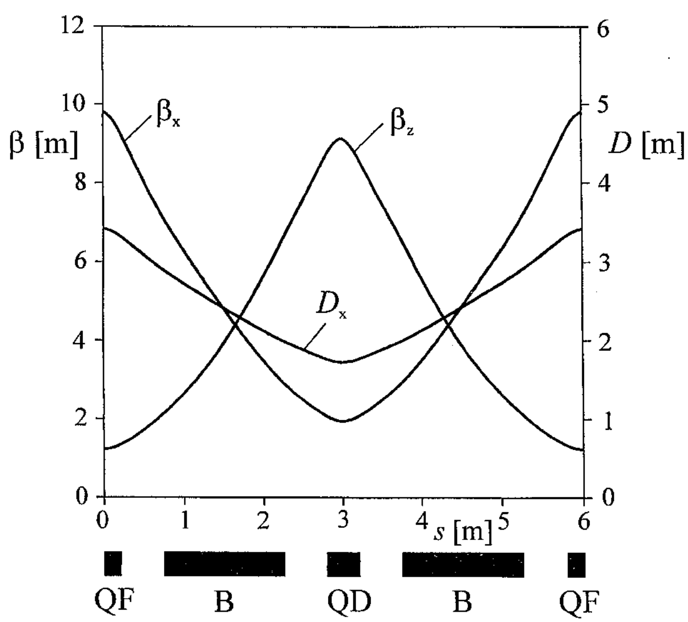
The FODO Cell

Alternating gradients allow focusing in both transverse plains. Therefore focusing and defocusing quadrupoles are usually alternated and interleaved with bending magnets.







$$\underline{M}_0 = \underline{M}_{FoDo}^N$$

The periodic beta function and dispersion for each FODO is also periodic for the whole ring. Usually only large sections of the ring consist of FODOs.

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Thin Lens FODO Cell

$$\underline{M} \approx \underline{Q}^{\text{thin}}(\tfrac{kl}{2})\underline{D}(\tfrac{L}{2})\underline{Q}^{\text{thin}}(-kl)\underline{D}(\tfrac{L}{2})\underline{Q}^{\text{thin}}(\tfrac{kl}{2})$$

$$=\begin{pmatrix}1 & 0\\ -\frac{kl}{2} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{L}{2}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{kl}{2} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{kl}{2} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{L}{2}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{kl}{2} & 1\end{pmatrix}=\begin{pmatrix}1+\frac{kl}{2}\frac{L}{2} & \frac{L}{2}\\ -(\frac{kl}{2})^2\frac{L}{2} & 1-\frac{kl}{2}\frac{L}{2}\end{pmatrix}\begin{pmatrix}1-\frac{kl}{2}\frac{L}{2} & \frac{L}{2}\\ -(\frac{kl}{2})^2\frac{L}{2} & 1+\frac{kl}{2}\frac{L}{2}\end{pmatrix}$$

$$=\begin{pmatrix}1-2(\frac{kl}{2}\frac{L}{2})^2 & L(1+\frac{kl}{2}\frac{L}{2})\\ -(\frac{kl}{2})^2 L(1-\frac{kl}{2}\frac{L}{2}) & 1-2(\frac{kl}{2}\frac{L}{2})^2\end{pmatrix} \Rightarrow$$

$$\cos\mu_{FODO} = 1-2(\tfrac{kl}{2}\tfrac{L}{2})^2 \qquad \beta = \left|\tfrac{L}{2\xi}\right|\sqrt{\tfrac{1+\xi}{1-\xi}} \qquad \alpha = 0$$

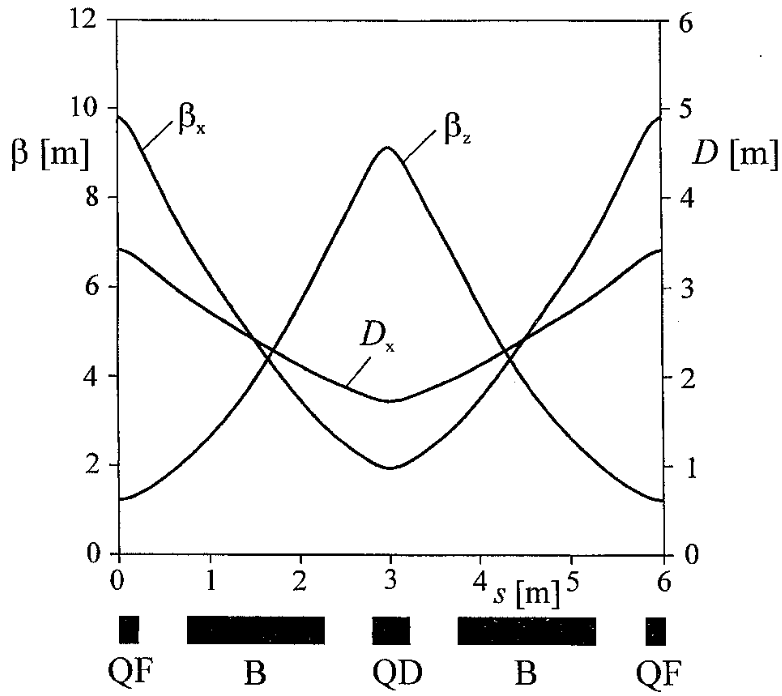
$$\Rightarrow \xi = \tfrac{kl}{2}\tfrac{L}{2}, \quad \sin\tfrac{\mu_{FODO}}{2} = |\xi|$$

$$\xi = \frac{kl}{2}\frac{L}{2}$$

$$\sin\frac{\mu_{FODO}}{2} = |\xi|$$

$$\beta = \left|\frac{L}{2\xi}\right|\sqrt{\frac{1+\xi}{1-\xi}}$$

$$\alpha = 0$$



$L_{FoDo} \approx 6\text{m}, \quad \varphi \approx 22.5°, \quad \mu_{FoDo} \approx \frac{\pi}{2}$

$\overline{\beta} \approx 3.8\text{m}$

$\beta_{\max} \approx 10.2\text{m}, \quad \beta_{\min} \approx 1.8\text{m}$

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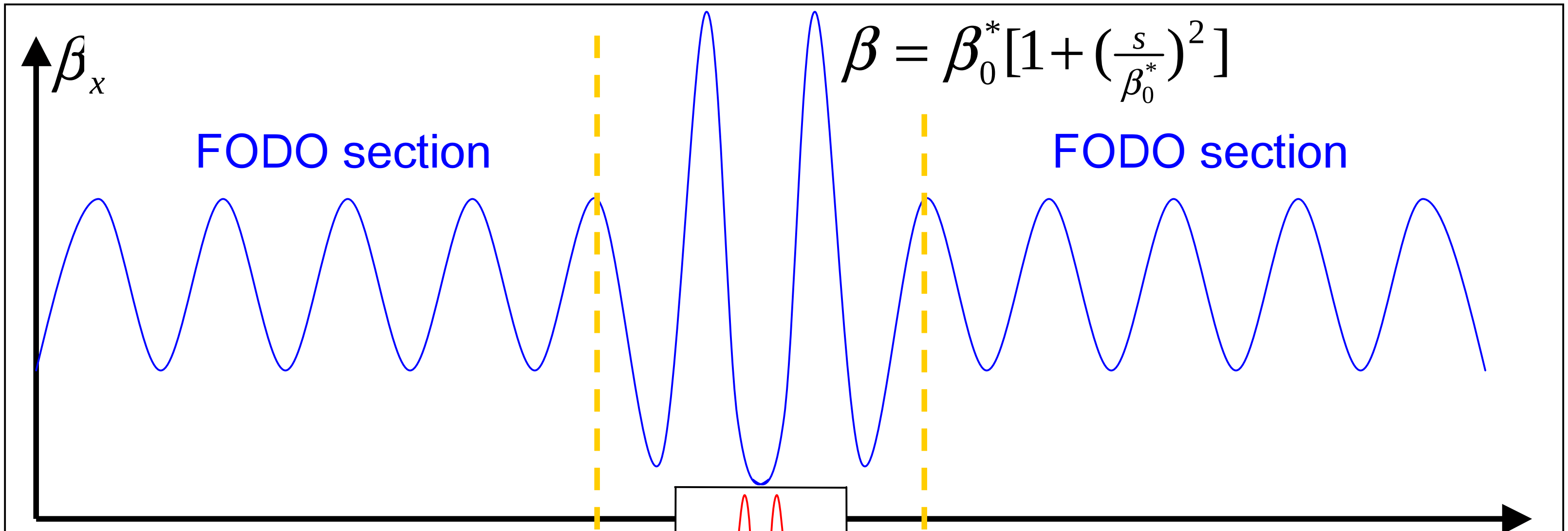
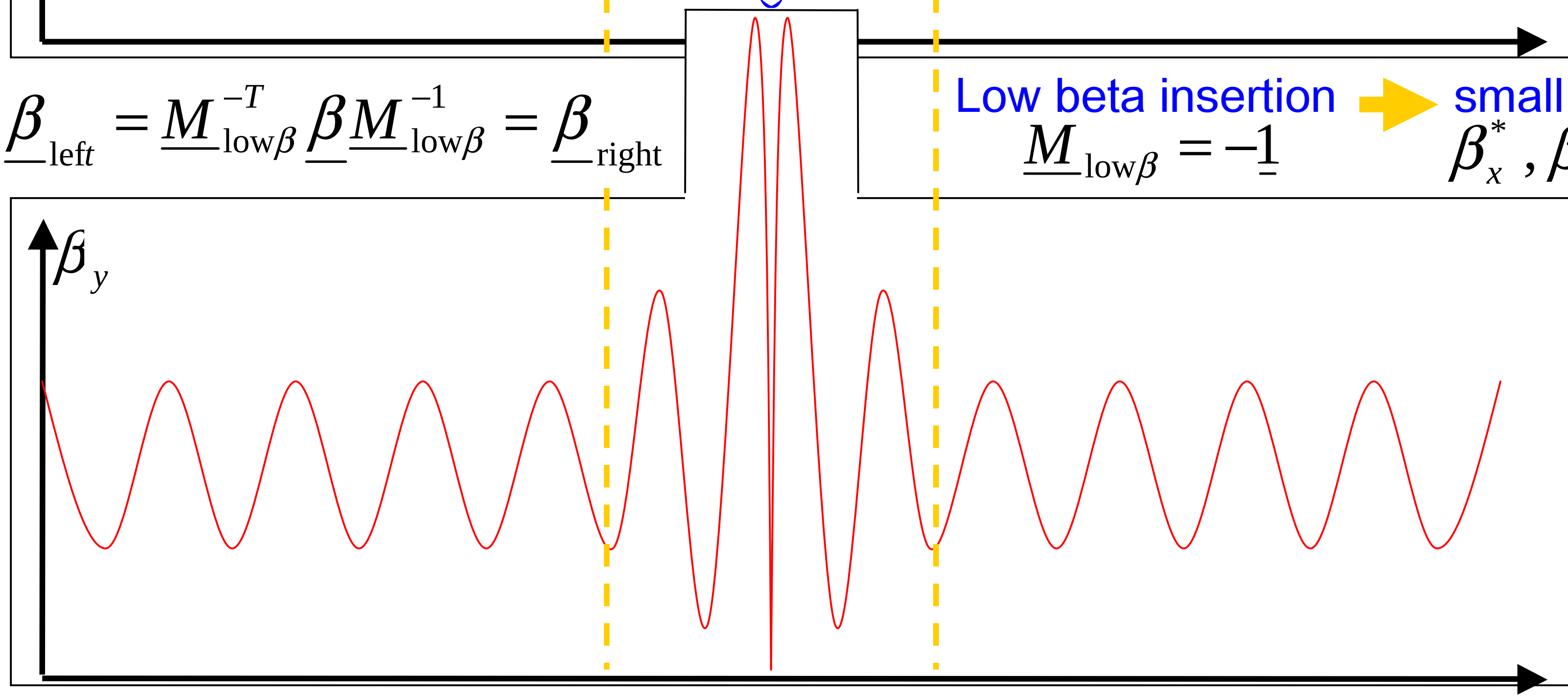
The Low Beta Insertion



$$\beta = \beta_0^*[1+(\frac{s}{\beta_0^*})^2]$$

$$\underline{\beta}_{\text{left}} = \underline{M}_{\text{low}\beta}^{-T}\,\underline{\beta}\,\underline{M}_{\text{low}\beta}^{-1} = \underline{\beta}_{\text{right}}$$

Low beta insertion $\underline{M}_{\text{low}\beta} = -\underline{1}$ ➜ small β_x^*, β_y^*



The Closed Orbit

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z}$ with $\Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$

For the periodic or closed orbit: $\vec{z}_{co} = \underline{M}_0\vec{z}_{co} + \underline{M}_0\int_0^L \underline{M}^{-1}(\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$

$$\vec{z}_{co} = [\underline{M}_0^{-1} - \underline{1}]^{-1}\int_0^L \underline{M}^{-1}(\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$= \frac{1}{2-2\cos\mu}[(\cos\mu - 1)\underline{1} + \sin\mu\,\underline{\beta}]\int_0^L \begin{pmatrix} -\sqrt{\beta\hat{\beta}}\sin\hat{\psi} \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\hat{\psi} + \alpha\sin\hat{\psi}] \end{pmatrix} \Delta\kappa(\hat{s})d\hat{s}$$

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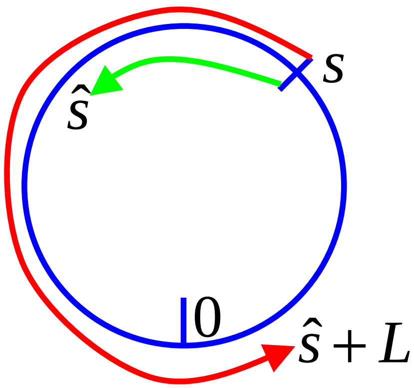
Closed Orbit Integral

$$x_{co}(0) = \frac{1}{2-2\cos\mu}\int_0^L \Delta\hat{\kappa}\sqrt{\beta\hat{\beta}}[(1-\cos\mu)\sin\hat{\psi}+\sin\mu\cos\hat{\psi}]d\hat{s}$$

$$= \frac{1}{4\sin^2\frac{\mu}{2}}\int_0^L \Delta\hat{\kappa}\sqrt{\beta\hat{\beta}}\,2\sin\tfrac{\mu}{2}[\sin\tfrac{\mu}{2}\sin\hat{\psi}+\cos\tfrac{\mu}{2}\cos\hat{\psi}]d\hat{s}$$

$$= \frac{\sqrt{\beta(0)}}{2\sin\frac{\mu}{2}}\int_0^L \Delta\kappa(\hat{s})\sqrt{\beta(\hat{s})}\cos(\psi(\hat{s})-\tfrac{\mu}{2})d\hat{s}$$

$$\cos\Big(\int_s^{\hat{s}\{+L\}}\tfrac{1}{\beta}d\hat{s}-\tfrac{\mu}{2}\Big) = \cos(\hat{\psi}-\psi\{+\mu\}-\tfrac{\mu}{2}) = \cos(|\hat{\psi}-\psi|-\tfrac{\mu}{2})$$



The {…} applies when $\hat{s}$ is smaller than s
and therefore $\hat{\psi}$ is smaller than ψ .

$$x_{co}(s) = \frac{\sqrt{\beta(s)}}{2\sin\frac{\mu}{2}}\oint \Delta\kappa(\hat{s})\sqrt{\beta(\hat{s})}\cos(|\psi(\hat{s})-\psi(s)|-\tfrac{\mu}{2})d\hat{s}$$

$$= \sum_k \Delta\vartheta_k \frac{\sqrt{\beta(s)\beta_k}}{2\sin\frac{\mu}{2}}\cos(|\psi_k-\psi|-\tfrac{\mu}{2})$$

Orbit from One Kick

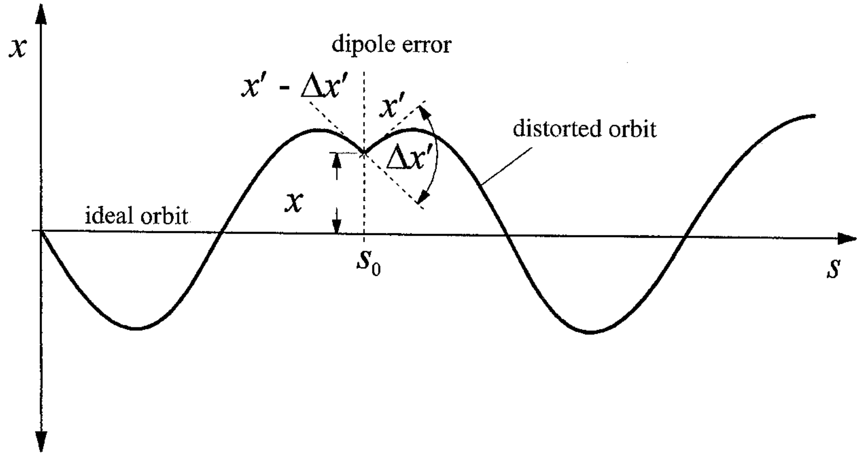
$$x_{co}(s) = \Delta\vartheta_k \frac{\sqrt{\beta(s)\beta_k}}{2\sin\frac{\mu}{2}} \cos(|\psi_k - \psi| - \tfrac{\mu}{2})$$

For $\psi > \psi_k$ this is a free betatron oscillation

$$x_{co}(s) = \Delta\vartheta_k \frac{\sqrt{\beta(s)\beta_k}}{2\sin\frac{\mu}{2}} \cos(\psi - \psi_k - \tfrac{\mu}{2})$$
$$= \sqrt{2J\beta(s)}\,\sin(\psi + \phi_0)$$
$$J = \Delta\vartheta_k^2 \frac{\beta_k}{8\sin^2\frac{\mu}{2}}, \quad \phi_0 = \frac{\pi}{2} - \psi_k - \frac{\mu}{2}$$

For $\psi \le \psi_k$ this is a free betatron oscillation

$$x_{co}(s) = \Delta\vartheta_k \frac{\sqrt{\beta(s)\beta_k}}{2\sin\frac{\mu}{2}} \cos(\psi - \psi_k + \tfrac{\mu}{2})$$
$$J = \Delta\vartheta_k^2 \frac{\beta_k}{8\sin^2\frac{\mu}{2}}, \quad \phi_0 = \frac{\pi}{2} - \psi_k + \frac{\mu}{2}$$



The oscillation amplitude J diverges when the tune ν is close to an integer.

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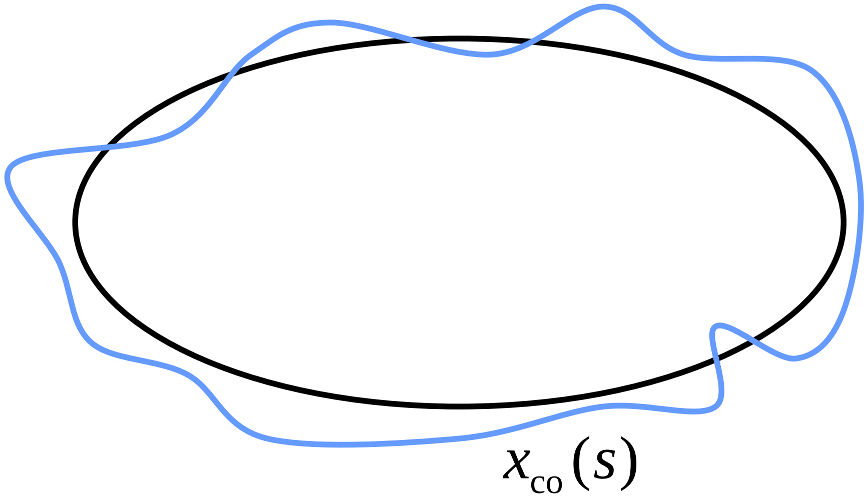
Oscillations around a Closed Orbit



Particles oscillate around this periodic orbit, not around the design orbit.

$$\vec{z} = \vec{z}_\beta + \vec{z}_{co}$$

$$\vec{z}_\beta(L) + \vec{z}_{co}(L) = \vec{z}(L) = \underline{M}_0 \vec{z}(0) + \Delta\vec{z} = \underline{M}_0[\vec{z}_\beta(0) + \vec{z}_{co}(0)] + \Delta\vec{z}$$

$$= \underline{M}_0 \vec{z}_\beta(0) + \vec{z}_{co}(L)$$

$$\boxed{\vec{z}_\beta(L) = \underline{M}_0 \vec{z}_\beta(0)}$$

The closed orbit does not change the linear transport matrix.

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Closed Orbit Correction

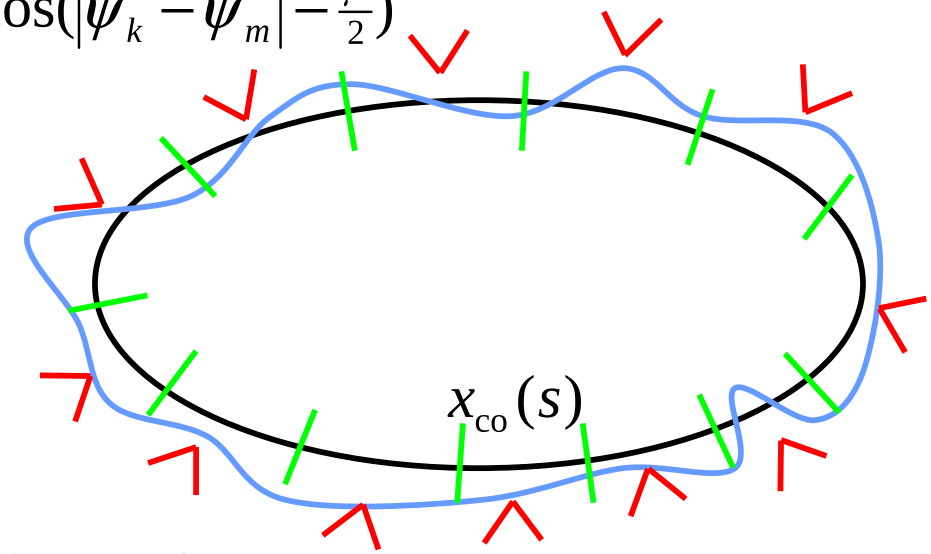
When the closed orbit $x_{co}^{old}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{co}^{new}(s_m) = x_{co}^{old}(s_m) + \sum_k \Delta\vartheta_k \frac{\sqrt{\beta_m\beta_k}}{2\sin\frac{\mu}{2}}\cos(|\psi_k - \psi_m| - \tfrac{\mu}{2})$$

$$= x_{co}^{old}(s_m) + \sum_k O_{mk}\Delta\vartheta_k$$

$$\vec{x}_{co}^{new} = \vec{x}_{co}^{old} = \underline{O}\Delta\vec{\vartheta}$$

$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{co}^{old} \quad \Rightarrow \quad \vec{x}_{co}^{new} = 0$$



It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit inbetween BPMs

Dispersion Integral

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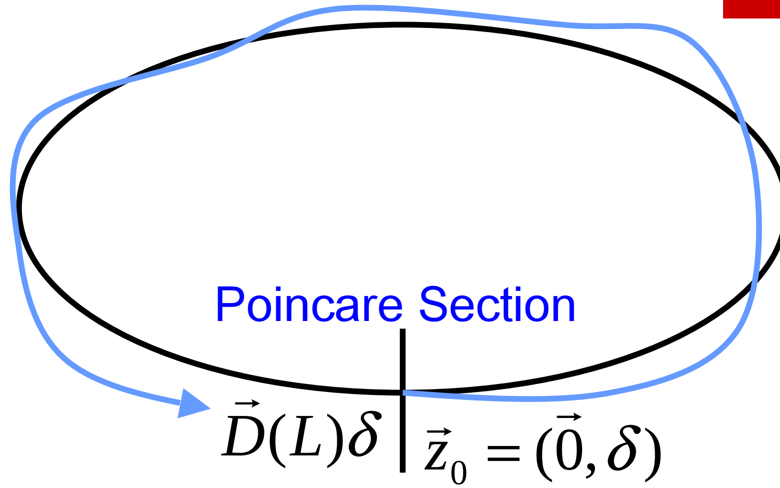
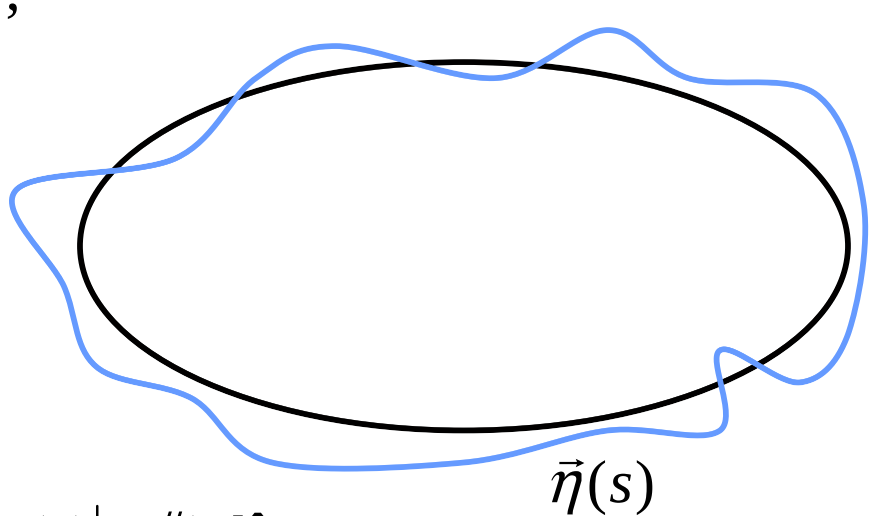
$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$



$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$\Rightarrow \quad \vec{D}(L) = \int_0^L \underline{M}(L-\hat{s})\begin{pmatrix} 0 \\ \kappa(\hat{s}) \end{pmatrix} ds'$$



$$\Delta\kappa = \delta\kappa$$

$$\eta(s) = \frac{\sqrt{\beta(s)}}{2\sin\frac{\mu}{2}} \oint \kappa(\hat{s})\sqrt{\beta(\hat{s})}\cos(|\psi(\hat{s}) - \psi(s)| - \tfrac{\mu}{2})d\hat{s}$$

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The Periodic Dispersion

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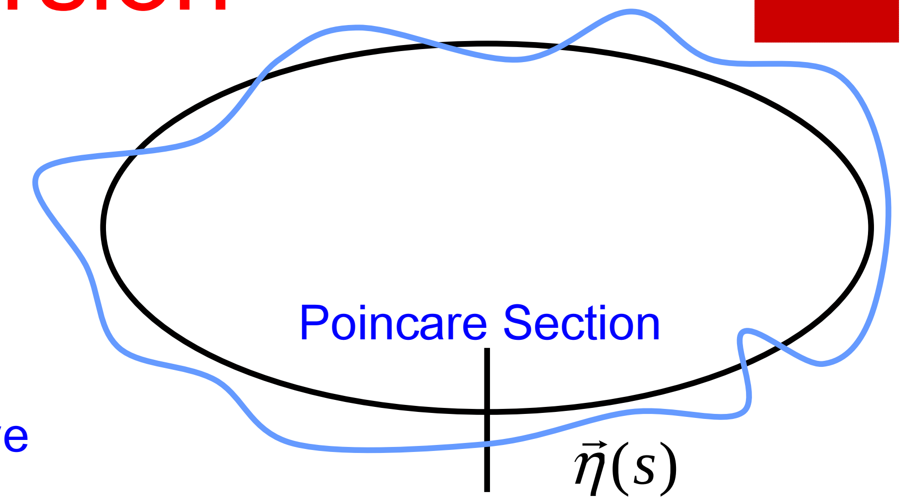
$$\begin{pmatrix} \vec{D}(L)\delta \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{M}_{0x} & \vec{0} & \vec{D}(L) \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{0} \\ 0 \\ \delta \end{pmatrix}$$



The periodic orbit for particles with relative energy deviation δ is

$$\vec{\eta}(L) = \underline{M}_{0x}\vec{\eta}(0) + \vec{D}(L) \quad \text{with} \quad \vec{\eta}(L) = \vec{\eta}(0)$$

$$\vec{\eta}(0) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L)$$

$$\Downarrow$$

$$\vec{\eta}(0) = [\underline{1} - \underline{M}_0(0)]^{-1}\vec{D}(L)$$

Particles with energy deviation δ is oscillate not around this periodic orbit.

$$\vec{z} = \vec{z}_\beta + \delta\vec{\eta}$$

$$\vec{z}_\beta(L) + \delta\vec{\eta}(L) = \vec{z}(L) = \underline{M}_0\vec{z}(0) + \vec{D}(L) = \underline{M}_0[\vec{z}_\beta(0) + \delta\vec{\eta}(0)] + \vec{D}(L)$$

$$= \underline{M}_0\vec{z}_\beta(0) + \delta\vec{\eta}(L)$$

Thin Lens FODO Cell

$$\underline{M} \approx \underline{Q}^{\text{thin}}(\tfrac{kl}{2})\underline{D}(\tfrac{L}{4})[\vec{\varphi} + \underline{Q}^{\text{thin}}(-kl)\underline{D}(\tfrac{L}{4})\vec{\varphi}]$$

$$\vec{D} = \begin{pmatrix} 1 & 0 \\ -\frac{kl}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{4} \\ 0 & 1 \end{pmatrix}\left\{\begin{pmatrix} 1 & \frac{L}{4} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{kl}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{kl}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{4} \\ 0 & 1 \end{pmatrix} + \underline{1}\right\}\begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{kl}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{4} \\ 0 & 1 \end{pmatrix}\left\{\begin{pmatrix} 1+\frac{kl}{2}\frac{L}{2} & \frac{L}{2}(1+\frac{kl}{2}\frac{L}{4}) \\ kl & 1+\frac{kl}{2}\frac{L}{2} \end{pmatrix} + \underline{1}\right\}\begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

$$= \begin{pmatrix} 2+\frac{kl}{2}L & L(1+\frac{1}{2}\frac{kl}{2}\frac{L}{2}) \\ -(\frac{kl}{2})^2 L & 2-\frac{kl}{2}\frac{L}{2}-(\frac{kl}{2}\frac{L}{2})^2 \end{pmatrix}\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} L(1+\frac{1}{2}\frac{kl}{2}\frac{L}{2}) \\ 2-\frac{kl}{2}\frac{L}{2}-(\frac{kl}{2}\frac{L}{2})^2 \end{pmatrix}\varphi$$

$$\vec{\eta} = [\underline{1}-\underline{M}]^{-1}\vec{D} = \frac{1}{4(\frac{kl}{2}\frac{L}{2})^2}\begin{pmatrix} 2(\frac{kl}{2}\frac{L}{2})^2 & L(1+\frac{kl}{2}\frac{L}{2}) \\ -(\frac{kl}{2})^2 L(1-\frac{kl}{2}\frac{L}{2}) & 2(\frac{kl}{2}\frac{L}{2})^2 \end{pmatrix}\begin{pmatrix} L(1+\frac{1}{2}\frac{kl}{2}\frac{L}{2}) \\ 2-\frac{kl}{2}\frac{L}{2}-(\frac{kl}{2}\frac{L}{2})^2 \end{pmatrix}\varphi$$

$$= L\frac{1+\frac{1}{2}\xi}{2\xi^2}\begin{pmatrix} \varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \eta \\ \eta' \end{pmatrix}$$

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FODO Example

$$\xi = \frac{kl}{2}\frac{L}{2}$$

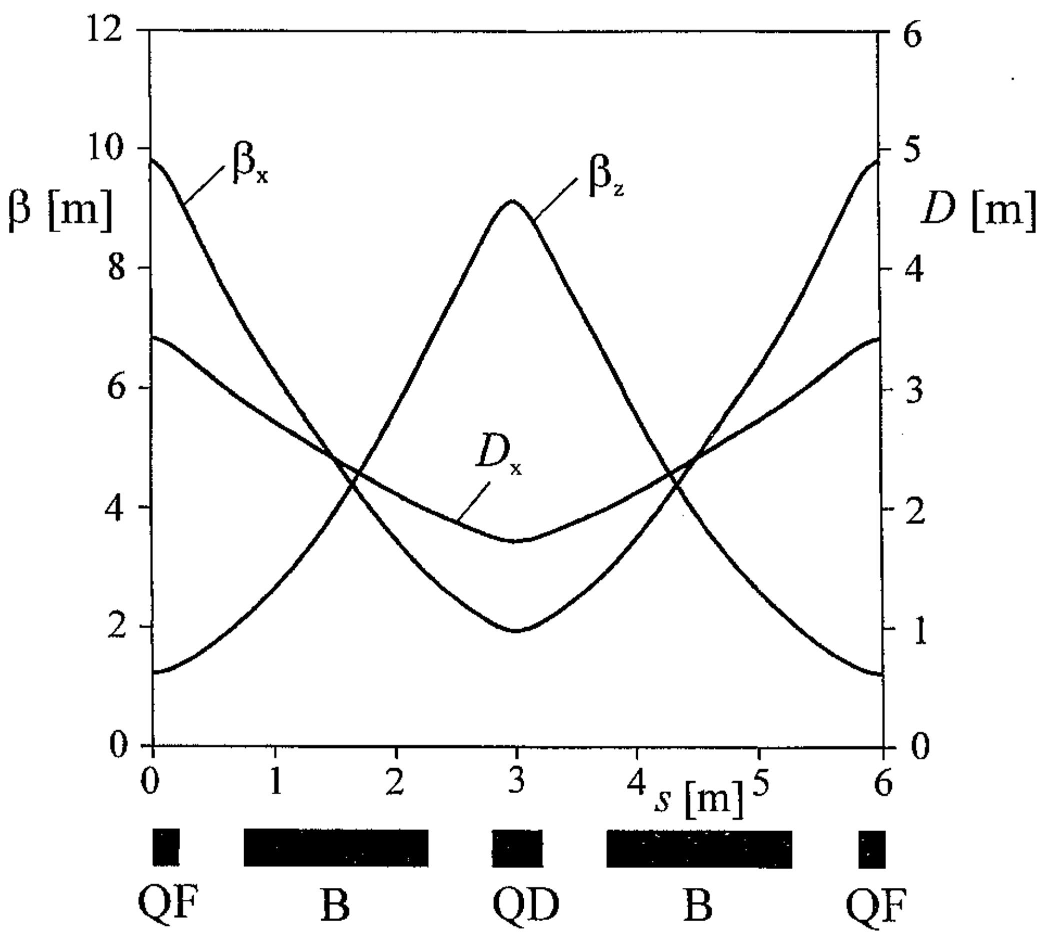
$$\sin\frac{\mu_{FODO}}{2} = |\xi|$$

$$\beta = \left|\frac{L}{2\xi}\right|\sqrt{\frac{1+\xi}{1-\xi}}$$

$$\alpha = 0$$

$$\eta = \frac{2+\xi}{(2\xi)^2}L\varphi$$

$$\eta' = 0$$



$L_{FoDo} \approx 6\text{m}\,,\quad \varphi \approx 22.5°\,,\quad \mu_{FoDo} \approx \frac{\pi}{2}$

$\overline{\beta} \approx 3.8\text{m}\,,\quad \overline{\eta} \approx 2\overline{\beta}\frac{\overline{\beta}}{\rho} \approx 3.8\text{m}$

$\beta_{\max} \approx 10.2\text{m}\,,\quad \beta_{\min} \approx 1.8\text{m}\,,\quad \eta_{\max} \approx 3.2\text{m}\,,\quad \eta_{\min} \approx 1.5\text{m}$

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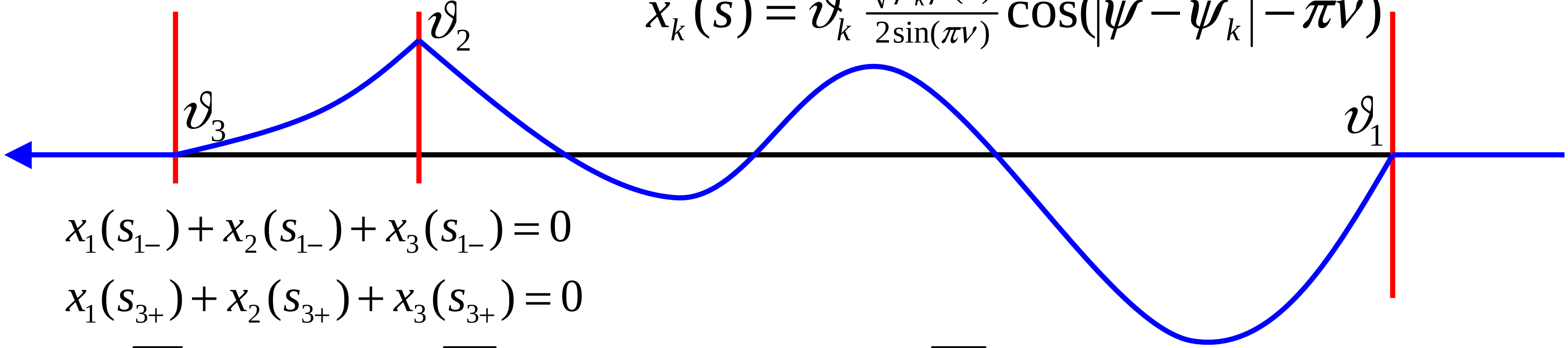
Closed Orbit Bumps

$$x_k(s)=\vartheta_k\frac{\sqrt{\beta_k\beta(s)}}{2\sin(\pi\nu)}\cos(|\psi-\psi_k|-\pi\nu)$$



$$x_1(s_{1-})+x_2(s_{1-})+x_3(s_{1-})=0$$

$$x_1(s_{3+})+x_2(s_{3+})+x_3(s_{3+})=0$$

$$\frac{\vartheta_1}{\vartheta_2}\sqrt{\beta_1}\cos(\pi\nu)+\frac{\vartheta_3}{\vartheta_2}\sqrt{\beta_3}\cos(|\psi_3-\psi_1|-\pi\nu)=-\sqrt{\beta_2}\cos(|\psi_2-\psi_1|-\pi\nu)$$

$$\frac{\vartheta_1}{\vartheta_2}\sqrt{\beta_1}\cos(|\psi_1-\psi_3|-\pi\nu)+\frac{\vartheta_3}{\vartheta_2}\sqrt{\beta_3}\cos(\pi\nu)=-\sqrt{\beta_2}\cos(|\psi_2-\psi_3|-\pi\nu)$$

$$\begin{pmatrix}\frac{\vartheta_1}{\vartheta_2}\\ \frac{\vartheta_3}{\vartheta_2}\end{pmatrix}=\frac{-\sqrt{\beta_2}}{N}\begin{pmatrix}\sqrt{\frac{1}{\beta_1}}\cos(\pi\nu) & -\sqrt{\frac{1}{\beta_1}}\cos(\psi_{31}-\pi\nu)\\ -\sqrt{\frac{1}{\beta_3}}\cos(\psi_{31}-\pi\nu) & \sqrt{\frac{1}{\beta_3}}\cos(\pi\nu)\end{pmatrix}\begin{pmatrix}\cos(\psi_{21}-\pi\nu)\\ \cos(\psi_{32}-\pi\nu)\end{pmatrix}$$

$$N=\cos^2(\pi\nu)-\cos^2(\psi_{31}-\pi\nu)=\sin(\psi_{31}-2\pi\nu)\sin\psi_{31}$$

$$\begin{pmatrix}\frac{\vartheta_1}{\vartheta_2}\\ \frac{\vartheta_3}{\vartheta_2}\end{pmatrix}=\frac{-1}{N}\begin{pmatrix}\sqrt{\frac{\beta_2}{\beta_1}}\sin(\psi_{31}-2\pi\nu)\sin\psi_{32}\\ \sqrt{\frac{\beta_2}{\beta_3}}\sin(\psi_{31}-2\pi\nu)\sin\psi_{21}\end{pmatrix}=\frac{-1}{\sin\psi_{31}}\begin{pmatrix}\sqrt{\frac{\beta_2}{\beta_1}}\sin\psi_{32}\\ \sqrt{\frac{\beta_2}{\beta_3}}\sin\psi_{21}\end{pmatrix}$$

$$\vartheta_1:\vartheta_2:\vartheta_3=\beta_1^{-\frac{1}{2}}\sin\psi_{32}:-\beta_2^{-\frac{1}{2}}\sin\psi_{31}:\beta_3^{-\frac{1}{2}}\sin\psi_{21}$$

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Quadrupole Errors

$$\vec{z}' = \underline{L}(s)\,\vec{z} + \Delta\vec{f}(\vec{z},s)$$

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z},\hat{s})\,d\hat{s} \approx \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_H,\hat{s})\,d\hat{s}$$

$$x'' = -(\kappa^2 + k)x - \Delta k(s)\,x \quad \Rightarrow \quad \begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} a \\ -(\kappa^2 + k)x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \Delta k(s) & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix}$$

$$\vec{z}(s) = \underline{M}(s)\vec{z}_0 - \int_0^s \underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta k(\hat{s}) & 0 \end{pmatrix}\underline{M}(\hat{s})\vec{z}_0\,d\hat{s}$$

$$\underline{M}_0(s) + \Delta\underline{M}_0(s) = \underline{M}_0(s) - \int_s^{s+L} \underline{M}(s+L,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta k(\hat{s}) & 0 \end{pmatrix}\underline{M}(\hat{s},s)\,d\hat{s}$$

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Quadrupole Error and Tune Shift

$$\underline{M}_0(s)+\Delta\underline{M}_0(s)=\underline{M}_0(s)-\int_s^{s+L}\underline{M}(s+L,\hat{s})\begin{pmatrix}0 & 0\\ \Delta k(\hat{s}) & 0\end{pmatrix}\underline{M}(\hat{s},s)d\hat{s}$$

$$\cos(\mu+\Delta\mu)=\cos\mu-\tfrac{1}{2}\int_s^{s+L}Tr\left[\underline{M}(s+L,\hat{s})\begin{pmatrix}0 & 0\\ \Delta k(\hat{s}) & 0\end{pmatrix}\underline{M}(\hat{s},s)\right]d\hat{s}$$

$$=\cos\mu-\tfrac{1}{2}\int_s^{s+L}Tr\left[\underline{M}_0(\hat{s})\begin{pmatrix}0 & 0\\ \Delta k(\hat{s}) & 0\end{pmatrix}\right]d\hat{s}$$

$$=\cos\mu-\tfrac{1}{2}\int_0^{L}\Delta k(\hat{s})\beta(\hat{s})\,d\hat{s}\sin\mu\approx\cos\mu-\Delta\mu\sin\mu$$

$$\Delta\mu=\tfrac{1}{2}\int_0^{L}\Delta k(\hat{s})\beta(\hat{s})\,ds$$

One quadrupole error:

$$\Delta\nu=\tfrac{\beta}{4\pi}\Delta k$$

→ More focusing always increases the tune

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Quadrupole Error and Beta Beat

$$\underline{M}_0(s)+\Delta\underline{M}_0(s)=\underline{M}_0(s)-\int_s^{s+L}\underline{M}(s+L,\hat{s})\begin{pmatrix}0 & 0\\ \Delta k(\hat{s}) & 0\end{pmatrix}\underline{M}(\hat{s},s)d\hat{s}$$

$$(\beta+\Delta\beta)\sin(\mu+\Delta\mu)=\beta\sin\mu-\int_s^{s+L}\Delta\hat{k}\beta\hat{\beta}\sin(\mu+\psi-\hat{\psi})\sin(\hat{\psi}-\psi)d\hat{s}$$

$$\approx\beta\sin\mu+\Delta\beta\sin\mu+\Delta\mu\beta\cos\mu$$

$$\Delta\beta=-\tfrac{1}{2\sin\mu}\int_s^{s+L}\Delta\hat{k}\beta\hat{\beta}[2\sin(\mu+\psi-\hat{\psi})\sin(\hat{\psi}-\psi)+\cos\mu]d\hat{s}$$

$$=\tfrac{\beta}{2\sin\mu}\int_s^{s+L}\Delta\hat{k}\hat{\beta}\cos(2[\hat{\psi}-\psi]-\mu)d\hat{s}=-\tfrac{\beta}{2\sin\mu}\int_0^{L}\Delta\hat{k}\hat{\beta}\cos(2|\hat{\psi}-\psi|-\mu)d\hat{s}$$

One quadrupole error:

$$\Delta\beta=-\tfrac{\beta}{2\sin\mu}\Delta\hat{k}\hat{\beta}\cos(2|\hat{\psi}-\psi|-\mu)$$

Focusing can increase or decrease the beta function

$$\frac{\Delta\beta_{\max}}{\beta}=2\pi\frac{\Delta\nu}{\sin\mu}$$

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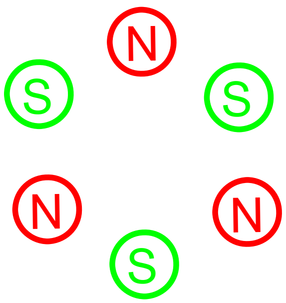
Sextupoles (revisited)

$$\psi = \Psi_3\,\mathrm{Im}\{(x-iy)^3\} = \Psi_3\cdot(y^3-3x^2y) \quad\Rightarrow\quad \vec{B} = -\vec{\nabla}\psi = \Psi_3\,3\begin{pmatrix}2xy\\ x^2-y^2\end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3\,3\begin{pmatrix}2xy\\ x^2-y^2\end{pmatrix}$$

$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3\,3\begin{pmatrix}2xy\\ x^2-y^2\end{pmatrix} + 6\Psi_3\Delta x\begin{pmatrix}y\\ x\end{pmatrix} + O(\Delta x^2)$$

$$k_2 = 3!\,\Psi_3 \Rightarrow k_1 = k_2\Delta x$$

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Chromaticity and its Correction

Chromaticity ξ = energy dependence of the tune

$$\upsilon(\delta) = \upsilon + \tfrac{\partial\upsilon}{\partial\delta}\delta + \dots$$

$$\xi = \tfrac{\partial\upsilon}{\partial\delta} \quad \text{with} \quad \upsilon = \tfrac{\mu}{2\pi}$$

Natural chromaticity ξ_0 = energy dependence of the tune due to quadrupoles only

$$\xi_{x0} = -\frac{1}{4\pi}\oint \beta_x(\hat{s})k_1(\hat{s})d\hat{s} \qquad \xi_{y0} = \frac{1}{4\pi}\oint \beta_y(\hat{s})k_1(\hat{s})d\hat{s}$$

Particles with energy difference oscillate around the periodic dispersion leading to a quadrupole effect in sextupoles that also shifts the tune:

$$\xi_x = \frac{1}{4\pi}\oint \beta_x(-k_1 + \eta_x k_2)d\hat{s} \qquad \xi_y = \frac{1}{4\pi}\oint \beta_y(k_1 - \eta_x k_2)d\hat{s}$$

Typically the the chormaticity ξ is chosen to be slightly positive, between 0 and 3.

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Nonlinear Motion

Sextupoles cause nonlinear dynamics, which can be chaotic and unstable.

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \underline{M}_0 \left[\begin{pmatrix} x_n \\ x'_n \end{pmatrix} - \frac{k_2 l_s}{2} \begin{pmatrix} 0 \\ x_n^{\ 2} \end{pmatrix} \right] \qquad \begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \hat{x}_n \\ \hat{x}'_n \end{pmatrix}$$

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \left[\begin{pmatrix} \hat{x}_n \\ \hat{x}'_n \end{pmatrix} - \frac{k_2 l_s}{2} \sqrt{\beta} \begin{pmatrix} 0 \\ \beta\, \hat{x}_n^2 \end{pmatrix} \right]$$

$$\begin{pmatrix} \hat{x}_f \\ \hat{x}'_f \end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \begin{pmatrix} 1-\cos\mu & \sin\mu \\ -\sin\mu & 1-\cos\mu \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \hat{x}_f^2 \end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \frac{1}{2\sin\frac{\mu}{2}} \begin{pmatrix} -\cos\frac{\mu}{2} \\ \sin\frac{\mu}{2} \end{pmatrix} \hat{x}_f^2$$

$$\left.\begin{aligned} \hat{x}_f &= -\frac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan\frac{\mu}{2} \\ \hat{x}'_f &= -\frac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan^2\frac{\mu}{2} \end{aligned}\right\} \hat{x} = \hat{x}_f + \Delta\hat{x} \qquad J_f = \frac{1}{2}(\hat{x}_f^2 + \hat{x}'^2_f) = \frac{1}{2\beta^3} \left(\frac{4}{k_2 l_s} \frac{\tan\frac{\mu}{2}}{\cos\frac{\mu}{2}} \right)^2$$

$$\begin{pmatrix} \Delta\hat{x}_{n+1} \\ \Delta\hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \left[\begin{pmatrix} \Delta\hat{x}_n \\ \Delta\hat{x}'_n \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \Delta\hat{x}_n^2 - 4\tan\frac{\mu}{2} \Delta\hat{x}_n \end{pmatrix} \right]$$

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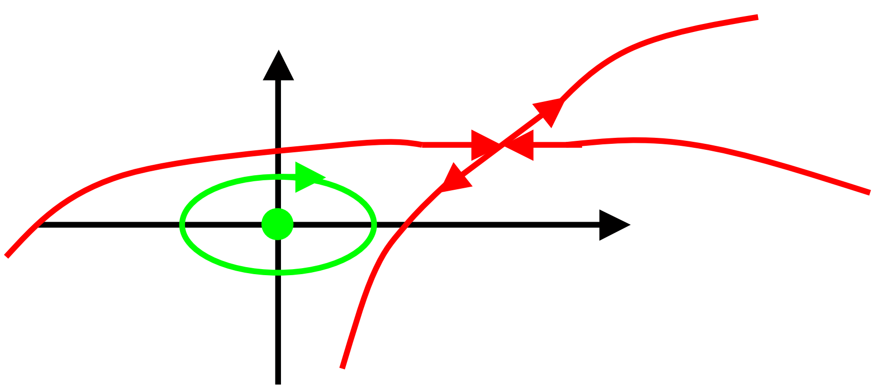
The Dynamic Aperture

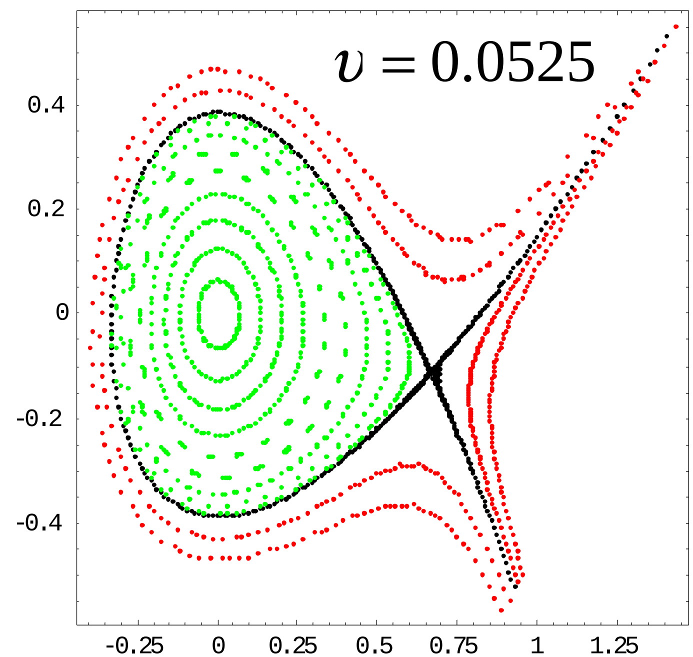
$$\begin{pmatrix}\Delta\hat{x}_{n+1}\\ \Delta\hat{x}'_{n+1}\end{pmatrix}=\begin{pmatrix}\cos\mu & \sin\mu\\ -\sin\mu & \cos\mu\end{pmatrix}\left[\begin{pmatrix}\Delta\hat{x}_{n}\\ \Delta\hat{x}'_{n}\end{pmatrix}-\begin{pmatrix}0\\ \frac{k_2 l_s}{2}\beta^{\frac{3}{2}}\Delta\hat{x}_n^2-4\tan\frac{\mu}{2}\Delta\hat{x}_n\end{pmatrix}\right]$$

$$\begin{pmatrix}\Delta\hat{x}_{n+1}\\ \Delta\hat{x}'_{n+1}\end{pmatrix}=\begin{pmatrix}\cos\mu+4\sin\mu\tan\frac{\mu}{2} & \sin\mu\\ -\sin\mu+4\cos\mu\tan\frac{\mu}{2} & \cos\mu\end{pmatrix}\left[\begin{pmatrix}\Delta\hat{x}_{n}\\ \Delta\hat{x}'_{n}\end{pmatrix}-\begin{pmatrix}0\\ \frac{k_2 l_s}{2}\beta^{\frac{3}{2}}\Delta\hat{x}_n^2\end{pmatrix}\right]$$

$$Tr[\underline{M}]=2\frac{\cos\frac{\mu}{2}(1+2\sin^2\frac{\mu}{2})}{\cos\frac{\mu}{2}}\geq 2$$

The additional fixed point is unstable !





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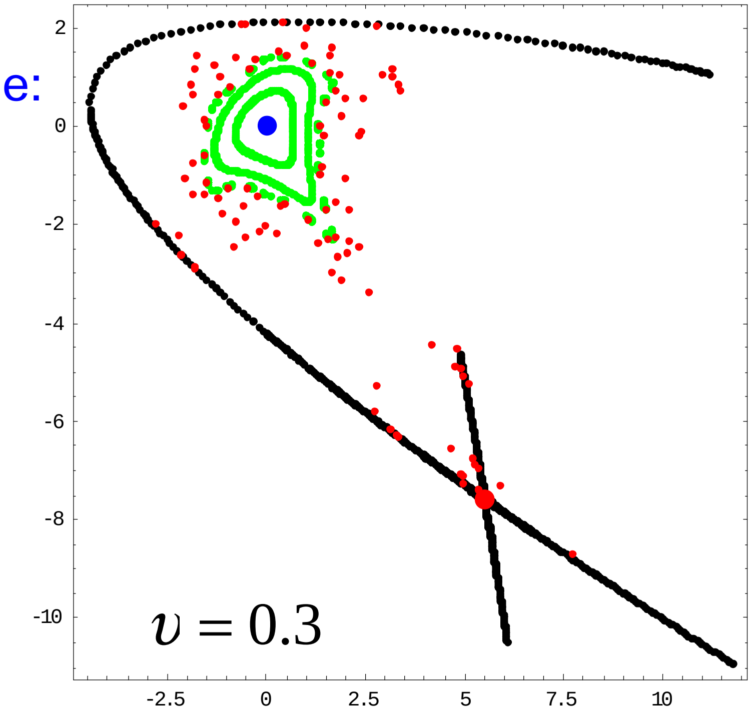
Sextupole Aperture

If the chormaticity is corrected by a single sextupole:

$$\xi_x = \xi_{0x} + \frac{1}{4\pi}\beta_x \eta_x k_2 l \approx 0$$

$$J_f = \frac{1}{2\beta^3}\left(\frac{4}{k_2 l_s}\frac{\tan\frac{\mu}{2}}{\cos\frac{\mu}{2}}\right)^2 \approx \frac{1}{2\beta}\left(\frac{\eta}{\xi_0 \pi}\frac{\sin\frac{\mu}{2}}{\cos^2\frac{\mu}{2}}\right)^2$$



Often the dynamic aperture is much smaller than the fixed point indicates !

When many sextupoles are used:

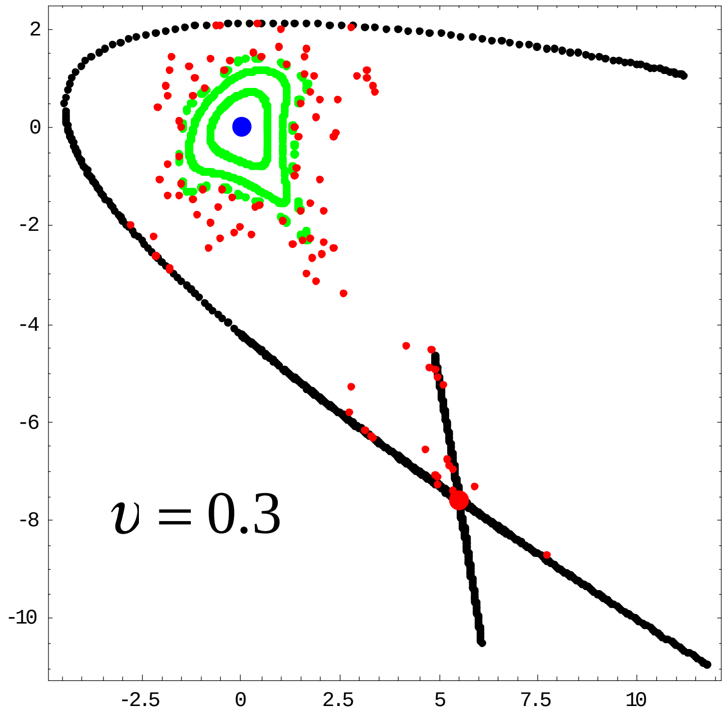
$$\xi_{0x} + \frac{N}{4\pi}\beta_x \eta_x k_2 l \approx 0$$

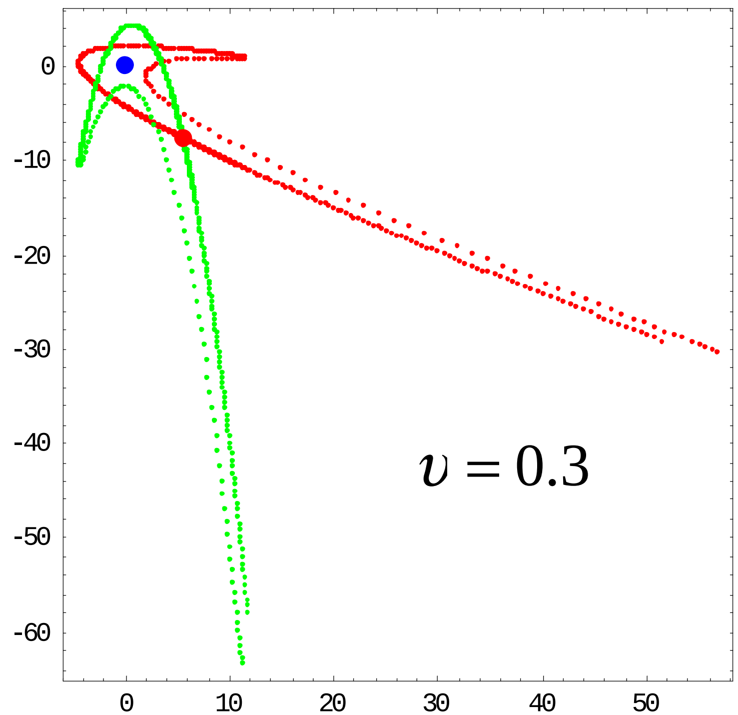
The sum of all k_2^2 is then reduced to about

$$\sum (k_2 l \beta)^2 \approx N(k_2 l\beta)^2 \approx \tfrac{1}{N}\left(\tfrac{4\pi}{\eta}\xi_0\right)^2$$

The dynamic aperture is therefore greatly increased when distributed sextupoles are used.

Sextupole Extraction



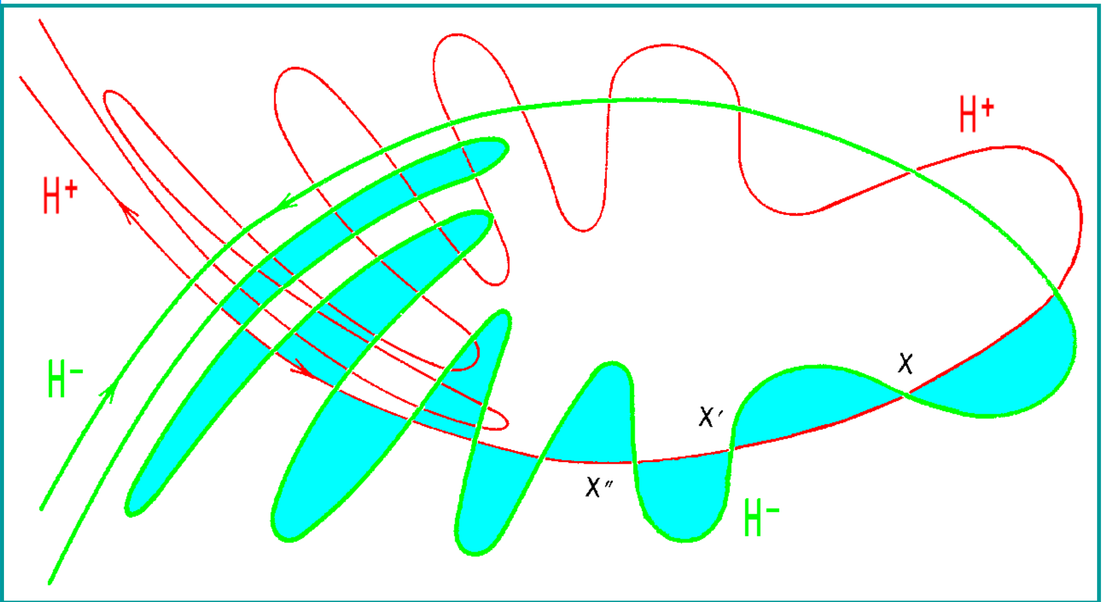


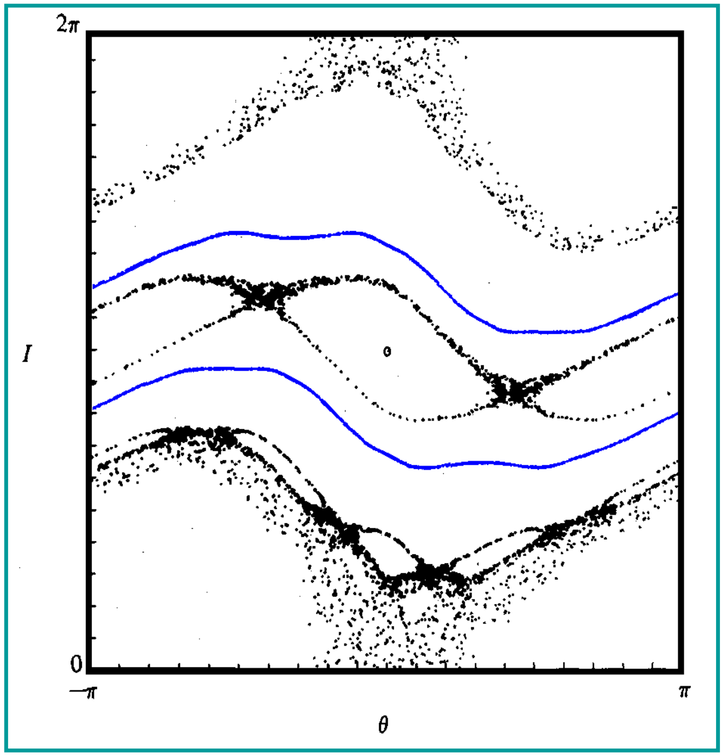
Due to the narrow region of unstable trajectories, sextupoles are used for slow particle extraction at a tune of 1/3.

The intersection of stable and unstable manifolds is a certain indication of chaos.

Homoclinic Points

- At instable fixed points, there is a stable and an instabile invariant curve.
- Intersections of these curves (homoclinic points) lead to chaos.





Perturbations

$$\begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} \begin{pmatrix} x \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi + \phi_0) \\ \cos(\psi + \phi_0) \end{pmatrix} = \sqrt{2J}\, \underline{\beta}\, \vec{S}$$

This would be a solution with constant J and ϕ when $\Delta f=0$.

Variation of constants:

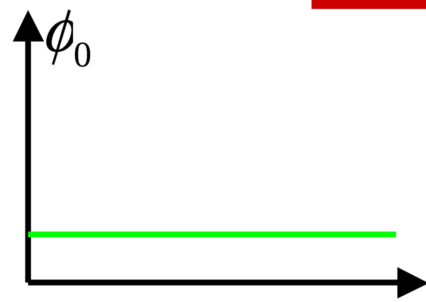
$$\frac{J'}{\sqrt{2J}} \underline{\beta}\, \vec{S} + \sqrt{2J}\ \phi_0' \begin{pmatrix} 0 & \sqrt{\beta} \\ -\frac{1}{\sqrt{\beta}} & -\frac{\alpha}{\sqrt{\beta}} \end{pmatrix} \vec{S} = \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\frac{J'}{\sqrt{2J}} \vec{S} + \sqrt{2J}\ \phi_0' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{S} = \underline{\beta}^{-1} \begin{pmatrix} 0 \\ \Delta f \end{pmatrix} \quad \text{with} \quad \underline{\beta}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

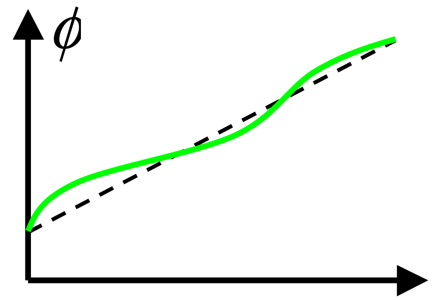
$$\frac{J'}{\sqrt{2J}} = \cos(\psi + \phi_0)\sqrt{\beta}\Delta f \quad , \quad \sqrt{2J}\ \phi_0' = -\sin(\psi + \phi_0)\sqrt{\beta}\Delta f$$

Simplification of linear motion

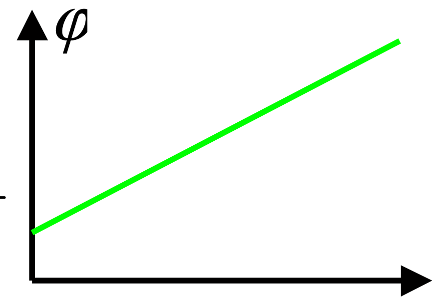
$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi + \phi_0) \\ \cos(\psi + \phi_0) \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \phi_0' = 0 \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \phi' = \frac{1}{\beta} \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi - \mu\frac{s}{L} + \varphi) \\ \cos(\psi - \mu\frac{s}{L} + \varphi) \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \varphi' = \mu\frac{1}{L} \end{matrix}$$



$$\tilde{\psi} = \psi - \mu\frac{s}{L} \Rightarrow \tilde{\psi}(s + L) = \tilde{\psi}(s)$$

Corresponds to Floquet's Theorem

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Quasi-periodic Perturbation

$$J' = \cos(\psi+\phi)\sqrt{2J\beta}\Delta f \quad , \quad \phi' = -\sin(\psi+\phi)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

$$J' = \cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\Delta f \quad , \quad \varphi' = \mu\tfrac{1}{L} - \sin(\tilde{\psi}+\varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

New independent variable $\vartheta = 2\pi\frac{s}{L}$

$$\tfrac{d}{d\vartheta}J = \cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\Delta f\,\tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi}+\varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f\,\tfrac{L}{2\pi}$$

$$\Delta f(x) = \Delta f(\sqrt{2J\beta}\sin(\tilde{\psi}+\varphi))$$

The perturbations are 2π periodic in ϑ and in φ

φ is approximately $\varphi \approx \upsilon \cdot \vartheta$

For irrational ν, the perturbations are quasi-periodic.

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Tune Shift with Amplitude

$$\tfrac{d}{d\vartheta} J = \cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\,\Delta f \,\tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi}+\varphi)\sqrt{\tfrac{\beta}{2J}}\,\Delta f\,\tfrac{L}{2\pi}$$

$$\tfrac{d}{d\vartheta}\varphi = \partial_J H \quad , \quad \tfrac{d}{d\vartheta} J = -\partial_\phi H \quad , \quad H(\varphi, J, \vartheta) = \upsilon \cdot J - \tfrac{L}{2\pi}\int_0^x \Delta f(\hat{x}, s)\, d\hat{x}$$

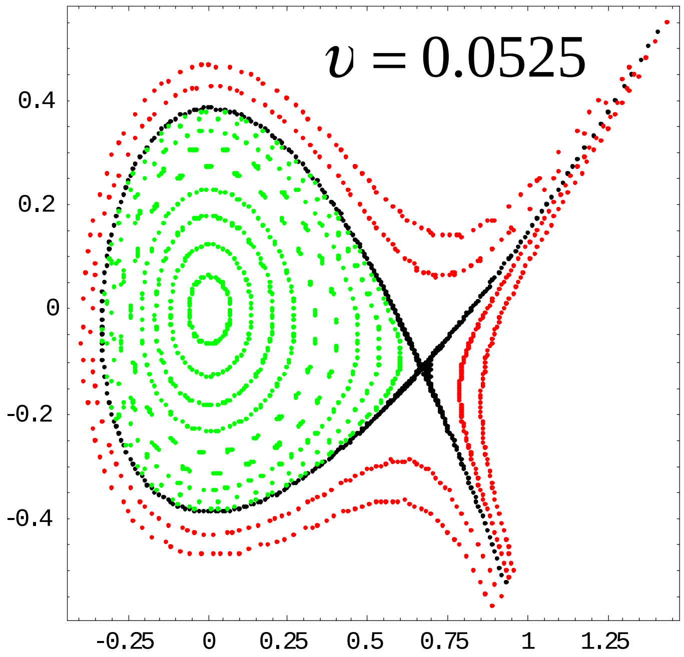
The motion remains Hamiltonian in the perturbed coordinates !

If there is a part in $\partial_J H$ that does not depend on φ, s $\Rightarrow$ Tune shift

The effect of other terms tends to average out.

$$\varphi(\vartheta) - \varphi_0 \approx \vartheta \cdot \partial_J \langle H \rangle_{\varphi,\vartheta}(J)$$

$$\upsilon(J) = \upsilon + \partial_J \langle \Delta H \rangle_{\varphi,\vartheta}(J)$$



Tune Shift Examples

$$H(\varphi, J) = \upsilon \cdot J - \tfrac{L}{2\pi}\int_0^x \Delta f(\hat{x}, s)\, d\hat{x} \quad , \quad \Delta\upsilon(J) = \partial_J \langle \Delta H \rangle_{\varphi,\vartheta}$$

Quadrupole: $\boxed{\Delta f = -\Delta k\, x}$

$$\Delta H = \tfrac{L}{2\pi}\Delta k \tfrac{1}{2} x^2 = \tfrac{L}{2\pi}\Delta k\, J\beta \sin^2(\tilde{\psi} + \varphi)$$

$$\langle \Delta H \rangle_{\varphi,\vartheta} = \tfrac{1}{2\pi}\int_0^{2\pi} \Delta k \beta\, d\vartheta\, L \frac{J}{4\pi} = \int_0^L \Delta k \beta\, ds \frac{J}{4\pi} \Rightarrow \boxed{\Delta\upsilon = \tfrac{1}{4\pi}\oint \Delta k \beta\, ds}$$

Sextupole: $\boxed{\Delta f = -k_2 \tfrac{1}{2} x^2}$

$$\Delta H = \tfrac{L}{2\pi} k_2 \tfrac{1}{3!} x^3 = \tfrac{L}{2\pi} k_2 \tfrac{1}{3!}\sqrt{2J\beta}^{\,3} \sin^3(\tilde{\psi} + \varphi)$$

$$\langle \Delta H \rangle_{\varphi,\vartheta} = 0 \quad \Rightarrow \quad \boxed{\Delta\upsilon = 0}$$

Octupole: $\boxed{\Delta f = -k_3 \tfrac{1}{3!} x^3}$

$$\Delta H = \tfrac{L}{2\pi} k_3 \tfrac{1}{4!} x^4 = \tfrac{L}{2\pi} k_3 \tfrac{1}{3!}(J\beta)^2 \sin^4(\tilde{\psi} + \varphi)$$

$$\langle \Delta H \rangle_{\varphi,\vartheta} = \tfrac{J^2}{3!2\pi}\oint k_3 \beta^2 ds \left\langle \tfrac{1}{2^4}(e^{i\varphi} - e^{-i\varphi})^4 \right\rangle_{\varphi} \Rightarrow \boxed{\Delta\upsilon = J\,\tfrac{1}{16\pi}\oint k_3 \beta^2 ds}$$

Nonlinear Resonances

$$\frac{d}{d\vartheta} J = \cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\Delta f \,\tfrac{L}{2\pi} \quad , \quad \frac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi}+\varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f \,\tfrac{L}{2\pi}$$

$$\frac{d}{d\vartheta}\varphi = \partial_J H \quad , \quad \frac{d}{d\vartheta} J = -\partial_\phi H \quad , \quad H(\varphi, J, \vartheta) = \upsilon \cdot J - \tfrac{L}{2\pi}\int_0^x \Delta f(\hat{x}, s)\, d\hat{x}$$

The effect of the perturbation is especially strong when

$\cos(\tilde{\psi}+\varphi)\sqrt{\beta}\Delta f$ or $\sin(\tilde{\psi}+\varphi)\sqrt{\beta}\Delta f$

has contributions that hardly change, i.e. the change of

$\sqrt{\beta(\vartheta)}\Delta f(x(\vartheta), \vartheta)$ is in resonance with the rotation angle $\varphi(\vartheta)$.

Periodicity allows Fourier expansion:

$$H(\varphi, J, \vartheta) = \sum_{n,m=-\infty}^{\infty} \widehat{H}_{nm}(J)\, e^{i[n\vartheta + m\varphi]} = \sum_{n,m=-\infty}^{\infty} H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$H_{00}(J) = \langle H(\varphi, J, s)\rangle_{\varphi,s} \Rightarrow$ Tune shift

The Single Resonance Model

$$\tfrac{d}{d\vartheta} J = \sum_{n,m=-\infty}^{\infty} m H_{nm}(J)\sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\tfrac{d}{d\vartheta}\varphi = \upsilon + \partial_J \sum_{n,m=-\infty}^{\infty} H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Strong deviation from: $J = J_0\ , \quad \varphi = \upsilon\vartheta + \varphi_0$

Occur when there is coherence between the perturbation and the phase space rotation: $n + m\tfrac{d}{ds}\varphi \approx 0$

Resonance condition: tune is rational $\boxed{n + m\ \upsilon = 0}$

On resonance the integral would increases indefinitely !

Neglecting all but the most important term

$$H(\varphi, J, \vartheta) \approx \upsilon J + H_{00}(J) + H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

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Fixed points

$$\frac{d}{d\vartheta} J = mH_{nm}(J)\sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\frac{d}{d\vartheta} \varphi = \upsilon + \Delta\upsilon(J) + \partial_J [H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))]$$

$$\Phi = \frac{1}{m}[n\vartheta + m\varphi + \Psi_{nm}(J)] \ , \quad \delta = \upsilon + \frac{n}{m}$$

$$\frac{d}{d\vartheta} J = mH_{nm}(J)\sin(m\Phi) \ , \quad \frac{d}{d\vartheta}\Phi = \delta + \Delta\upsilon(J) + H'_{nm}(J)\cos(m\Phi)$$

$$H(\varphi, J, \vartheta) \approx \delta\, J + H_{00}(J) + H_{nm}(J)\cos(m\Phi)$$

Fixed points: $\frac{d}{d\vartheta} J = mH_{nm}(J_f)\sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \frac{k}{m}\pi$

If $\delta + \Delta\upsilon(J_f) \pm H'_{nm}(J_f) = 0$ has a solution.

$$\frac{d}{d\vartheta}\Delta J = \pm m^2 H_{nm}(J_f)\Delta\Phi \ , \quad \frac{d}{d\vartheta}\Delta\Phi = [\Delta\upsilon'(J_f) \pm H''_{nm}(J_f)]\Delta J$$

Stable fixed point for: $H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta\upsilon'(J_f)] < 0$

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Third Integer Resonances

Sextupole: $\Delta f = -k_2 \frac{1}{2} x^2$

$$\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!} x^3 = \frac{L}{2\pi} k_2 \frac{1}{3!} \sqrt{2J\beta}^3 \sin^3(\tilde{\psi} + \varphi)$$

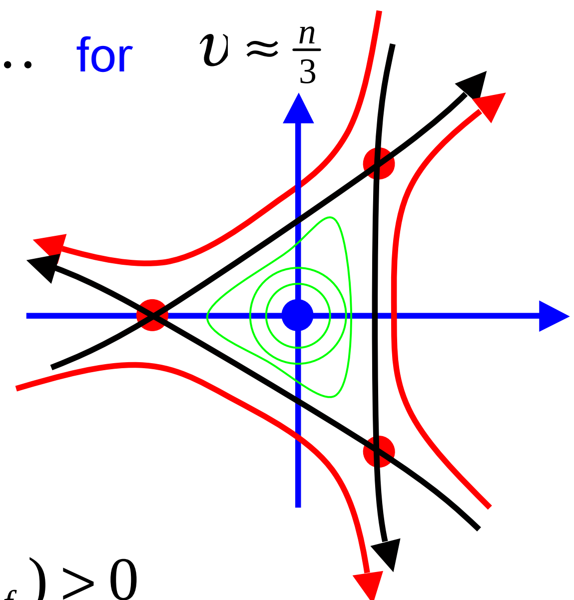
$$= \frac{L}{2\pi} k_2 \frac{1}{3!4} \sqrt{2J\beta}^3 [\sin(3[\tilde{\psi} + \varphi]) + 3\sin(\tilde{\psi} + \varphi)]$$

Simplification: one sextupole $\quad k_2(\vartheta) = k_2 \delta(\vartheta) = k_2 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n\vartheta)$

$$\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!4} \sqrt{2J\beta}^3 \frac{1}{2\pi} \cos(-n\vartheta + 3\varphi + \tilde{\psi} - \tfrac{\pi}{2}) + \ldots$$ for $\upsilon \approx \frac{n}{3}$

$$\Delta H \approx A_2 \sqrt{J}^3 \cos(3\Phi)$$

$$\left.\begin{array}{l} \Phi_f = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \ldots \\ \delta \pm A_2 \frac{3}{2}\sqrt{J} = 0 \end{array}\right\} \begin{array}{l} \Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi \\ \text{for } \delta > 0 \end{array}$$



All these fixed points are instable since $H_{nm}(J_f) H''_{nm}(J_f) > 0$

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Fourth Integer Resonances

Octupole: $\Delta f = -k_3 \frac{1}{3!} x^3$, $\Delta H = \frac{L}{2\pi} k_2 \frac{1}{4!} x^4 = \frac{L}{2\pi} k_2 \frac{1}{3!} J^2 \beta^2 \sin^4(\tilde{\psi} + \varphi)$

$$= \frac{L}{2\pi} k_2 \frac{1}{3!8} J^2 \beta^2 [\cos(4[\tilde{\psi} + \varphi]) - 4\cos(\tilde{\psi} + \varphi) + 6]$$

Simplification: one octupole $k_3(\vartheta) = k_3 \delta(\vartheta) = k_3 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n\vartheta)$

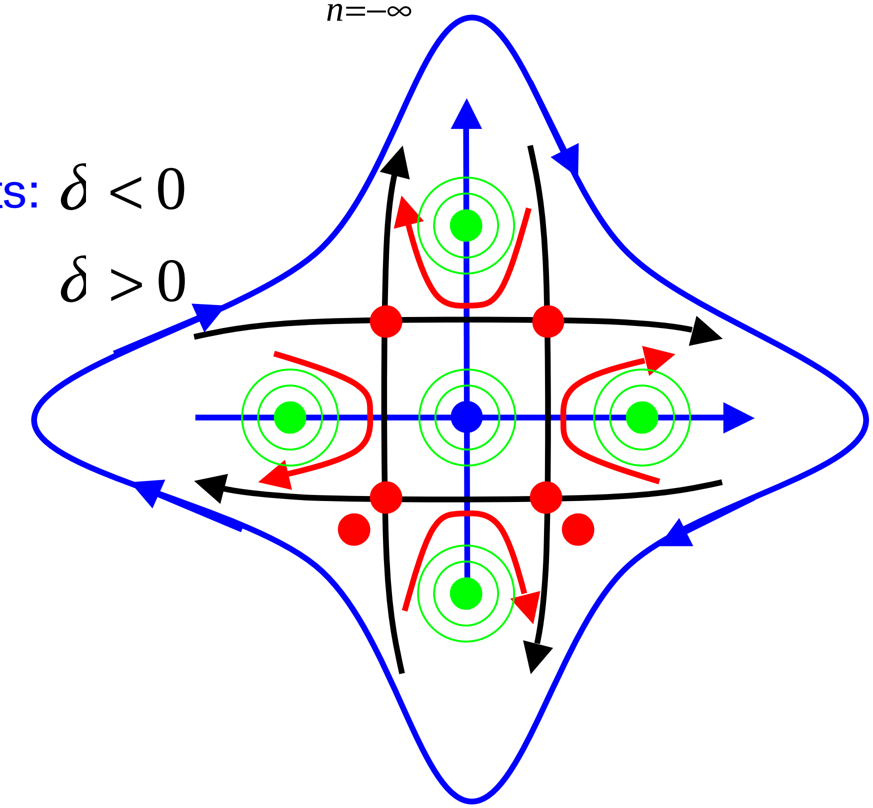
$\Delta H \approx A_3 J^2 [6 + \cos(4\Phi)]$ for $\upsilon \approx \frac{n}{4}$

$\Phi_f = 0, \frac{1}{4}\pi, \frac{2}{4}\pi, \ldots$ Either 8 fixed points: $\delta < 0$

$\delta + A_3 2J(6 \pm 1) = 0$ or none for: $\delta > 0$

$H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta\upsilon'(J_f)] < 0$

Stability for $(2A_3 J)^2 [1 \pm 6] < 0$,
i.e. for the 4 outer fixed points.



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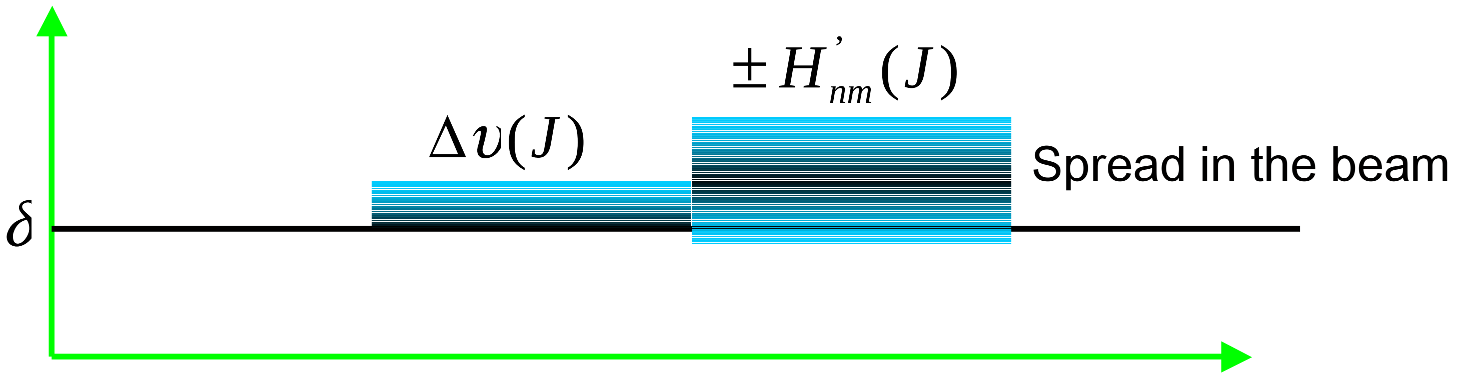
Resonance Width (Strength)

Fixed points: $\frac{d}{d\vartheta} J = m H_{nm}(J_f)\sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \frac{k}{m}\pi$

If $\delta + \Delta\upsilon(J_f) \pm H'_{nm}(J_f) = 0$ has a solution.

δ has to avoid the region $\delta + \Delta\upsilon(J) \pm H'_{nm}(J) = 0$ for all particles.



Assuming that the tune shift and perturbation are monotonous in J:

This tune region has the width $\Delta_{nm} = 2\,|\,H'_{nm}(J_{\max})|$ for strong resonances.

Δ_{nm} Is called Resonance Width, Resonance Strength, or Stop-Band Width

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Coupling Resonances

$$\frac{d}{d\vartheta} J_x = \cos(\tilde{\psi}_x + \varphi_x)\sqrt{2J_x\beta_x}\Delta f_x \frac{L}{2\pi} \quad , \quad \frac{d}{d\vartheta}\varphi_x = \upsilon_x - \sin(\tilde{\psi}_x + \varphi_x)\sqrt{\frac{\beta_x}{2J_x}}\Delta f_x \frac{L}{2\pi}$$

$$\frac{d}{d\vartheta} J_y = \cos(\tilde{\psi}_y + \varphi_y)\sqrt{2J_y\beta_y}\Delta f_y \frac{L}{2\pi} \quad , \quad \frac{d}{d\vartheta}\varphi_y = \upsilon_y - \sin(\tilde{\psi}_y + \varphi_y)\sqrt{\frac{\beta_y}{2J_y}}\Delta f_y \frac{L}{2\pi}$$

$$\frac{d}{d\vartheta}\vec{\varphi} = \vec{\partial}_J H \quad , \quad \frac{d}{d\vartheta}\vec{J} = -\vec{\partial}_\varphi H \quad , \quad H(\vec{\varphi}, \vec{J}, \vartheta) = \vec{\upsilon}\cdot\vec{J} - \frac{L}{2\pi}\int_0^{\vec{x}} \Delta\vec{f}(\hat{\vec{x}}, s)d\hat{\vec{x}}$$

The integral form can be chosen since it is path independent. This is due to the Hamiltonian nature of the force: $\Delta\vec{f}(x,y,s) = -\partial_{x,y}\Delta H(s,y,p_x,p_y,s)$

Single Resonance model for two dimensions means retaining only the amplitude dependent tune shift and one term in the two dimensional Fourier expansion:

$$H(\vec{\varphi}, \vec{J}, \vartheta) = \vec{\upsilon}\cdot\vec{J} + H_{00}(\vec{J}) + H_{n\vec{m}}(\vec{J})\cos(n\vartheta + m_x\varphi_x + m_y\varphi_y + \Psi_{n\vec{m}}(\vec{J}))$$

For $n + m_x\upsilon_x + m_y\upsilon_y \approx 0$

Sum and Difference Resonances

$n + m_x \upsilon_x + m_y \upsilon_y \approx 0$ means that oscillations in y can drive oscillations in x in

$$x'' = -K\,x + \Delta f(x,s)$$

The resonance term in the Hamiltonian then changes only slowly:

$$H(\vec{\varphi},\vec{J},\vartheta) = \vec{\upsilon}\cdot\vec{J} + H_{00}(\vec{J}) + H_{n\vec{m}}(\vec{J})\cos(n\vartheta + m_x\varphi_x + m_y\varphi_y + \Psi_{n\vec{m}}(\vec{J}))$$

$$\tfrac{d}{d\vartheta}\vec{\varphi} = \vec{\partial}_J H \quad , \quad \tfrac{d}{d\vartheta}\vec{J} = -\vec{\partial}_\varphi H$$

$$J = \vec{m}\cdot\vec{J}$$

$$J_\perp = m_x J_x - m_y J_y = \vec{m}\times\vec{J} \quad \Rightarrow \quad \tfrac{d}{d\vartheta} J_\perp = 0$$

Difference resonances lead to stable motion since:

$$n + |m_x|\upsilon_x - |m_y|\upsilon_y \approx 0 \Rightarrow |m_x|J_x + |m_y|J_y = const.$$

Sum resonances lead to unstable motion since:

$$n + |m_x|\upsilon_x + |m_y|\upsilon_y \approx 0 \Rightarrow |m_x|J_x - |m_y|J_y = const.$$

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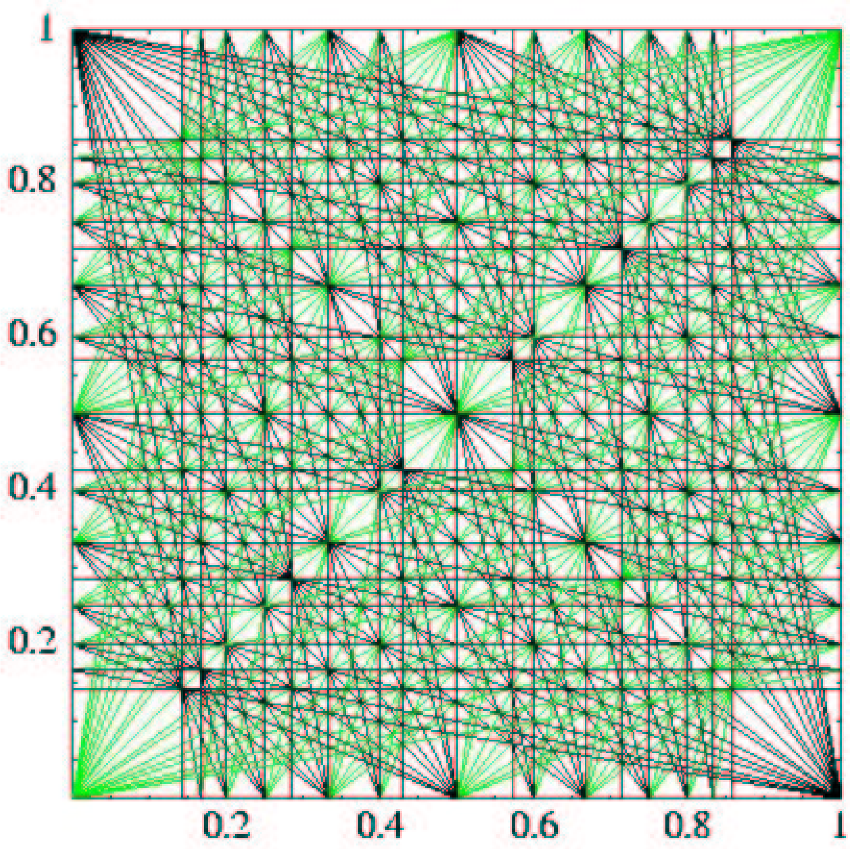
Resonances Diagram

$n + m_x \upsilon_x + m_y \upsilon_y \approx 0$ means that oscillations in y can drive oscillations in x in

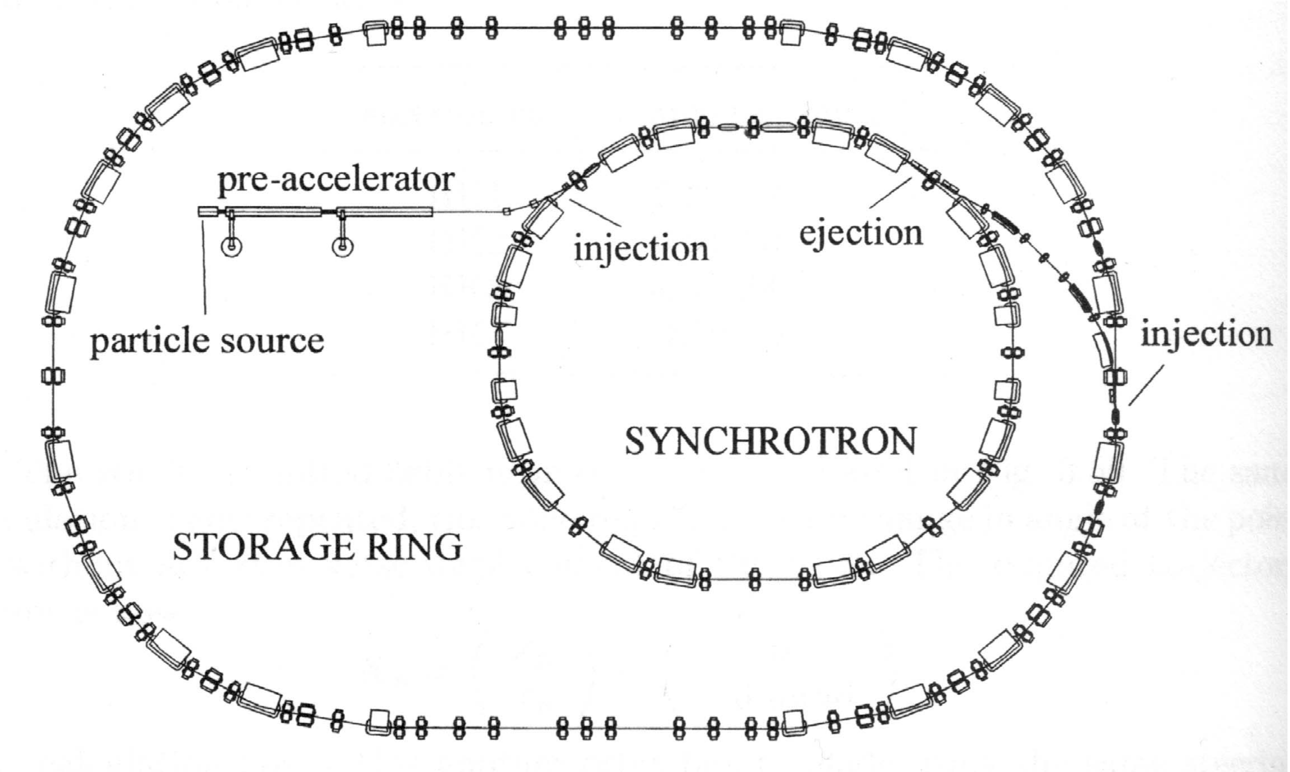
$$x'' = -K\,x + \Delta f\,(x, s)$$



All these resonances have to be avoided by their respective resonance width.

The position of an accelerator in the tune plane s called its Working Point.

Injection and Extraction



High energy accelerators are fed by a pre-accelerator chain. For each energy stage there is an appropriate accelerator.

Particle transfer from one accelerator to the other must have as few particle losses as possible.

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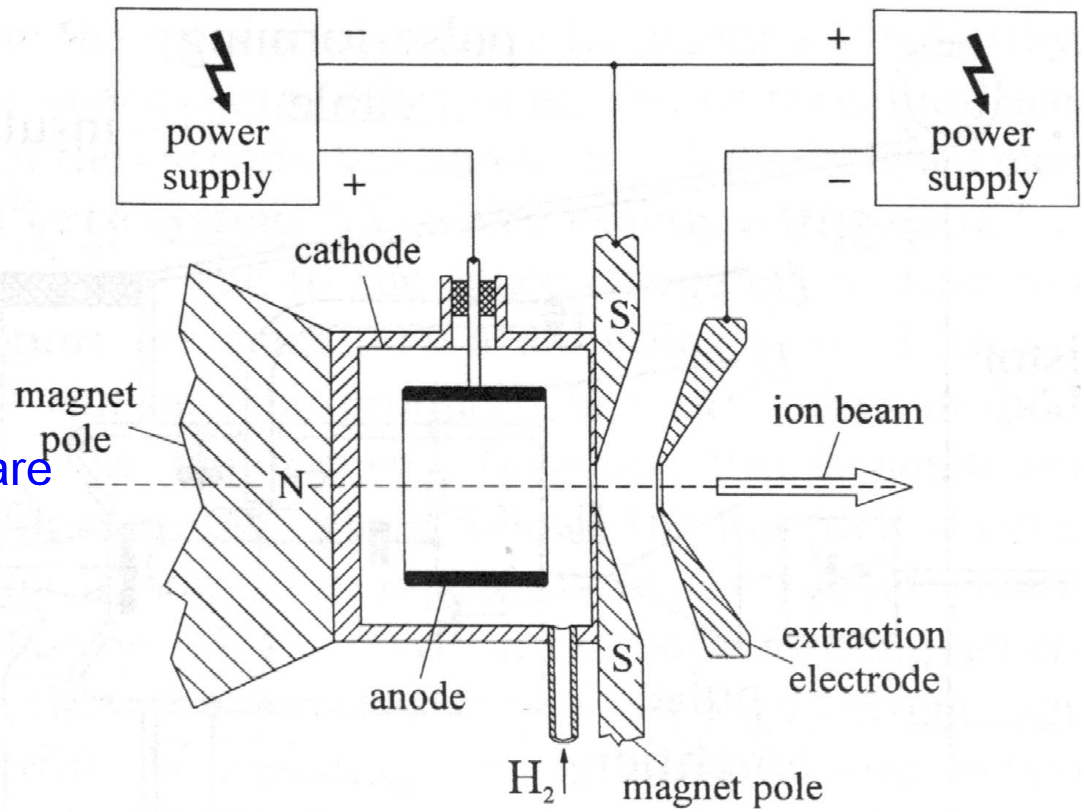
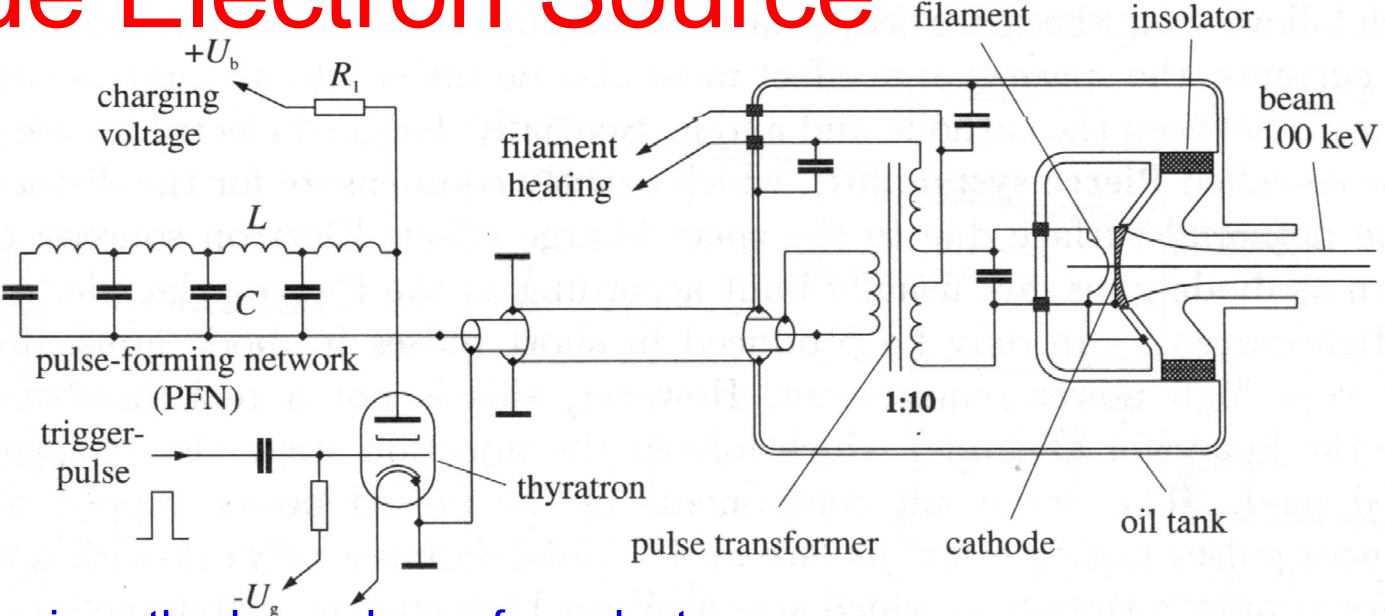
The PIG Ion Source

Penning Principle
(of the Philips Ion Gage)

- Magnetic field of about 0.01T.
- Pressurized gas is inserted
  at <100Pa (10-3Atm)
- Gas is ionized and remains
  ionized since electrons are
  accelerated in the E and
circle
  in the B-field.
- Positive ions are accelerated
  through a hole in the
cathode
  to several 100V.



Diode Electron Source



- A thermionic cathode produces free electrons.
- An earthed anode accelerates them through an aperture into a linac.
- The cathode is not flat but curved (Pierce Cathode) to produce a force that counters Coulomb expulsion (the Space Charge Force)
- Typical voltages are 100-150kV, typical peak currents are a few Ampere.
- Due to power limits, only short pulses can be produced (> a few μs long)
- A thyratron is used as fast high-current switch and capacitors provide the short pulse.
- The pulse from the capacitors is magnified (by about 10) in a transformer to reach the 100-150kV.

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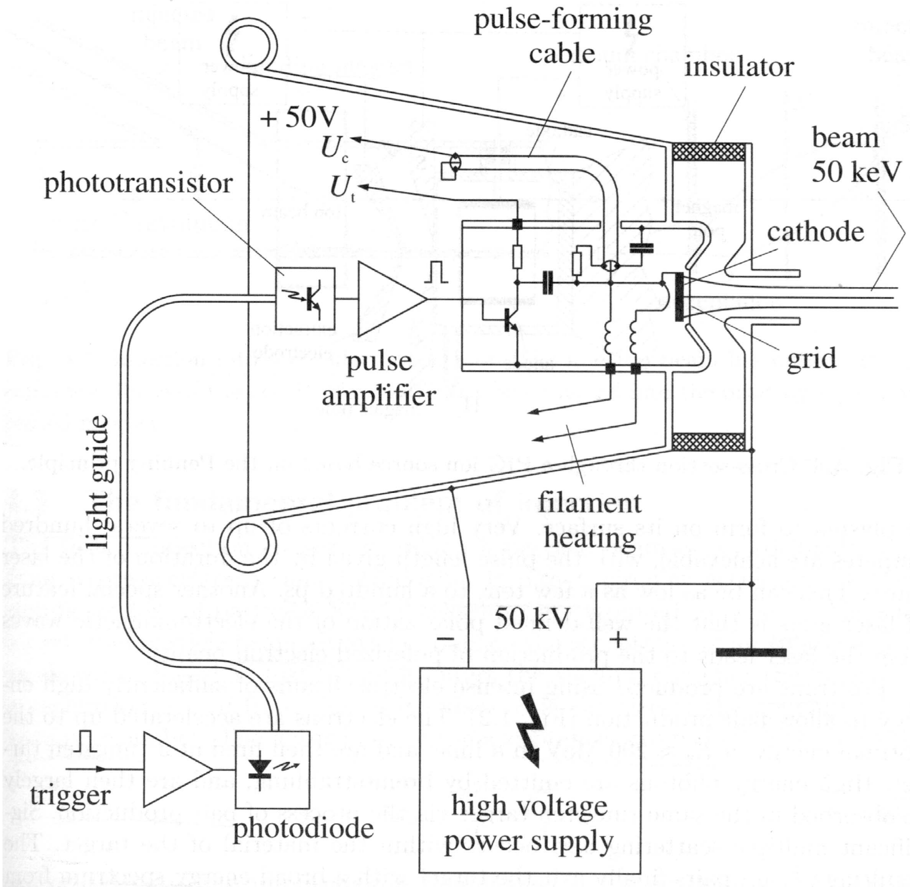
Triode Electron Source



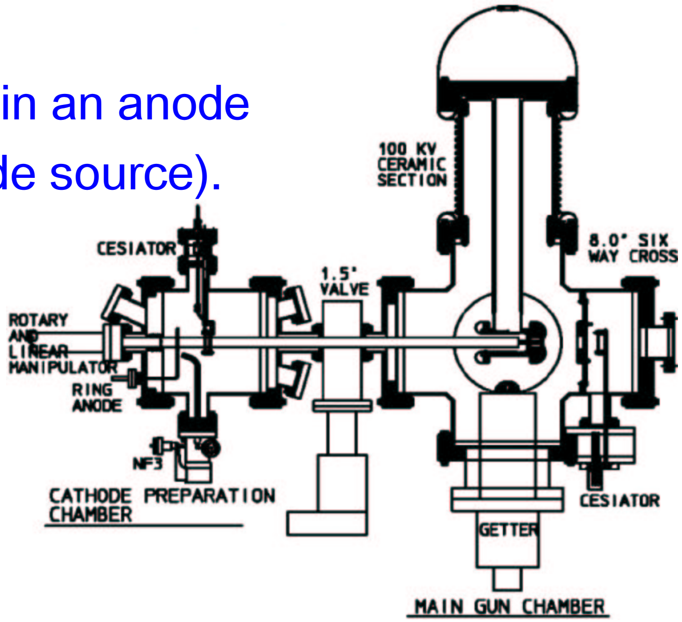
- There is no transformer and therefore pulses can be shorter (>1 ns long)
- A thermionic cathode produces free electrons.
- A 50V barrier grid prohibits electrons from leaving the cathode.
- An earthed anode accelerates them through an aperture into a linac.
- Typical voltages are 50kV, typical peak currents are a few Ampere.
- The short pulse amplifier is in a Faraday cage at high potential.

- Alight guide transports a short trigger pulse to high potential.
- The amplified pulse then only has to switch the 50V of the grid.

Other Electron and Positron Guns

Photo-Cathode Sources

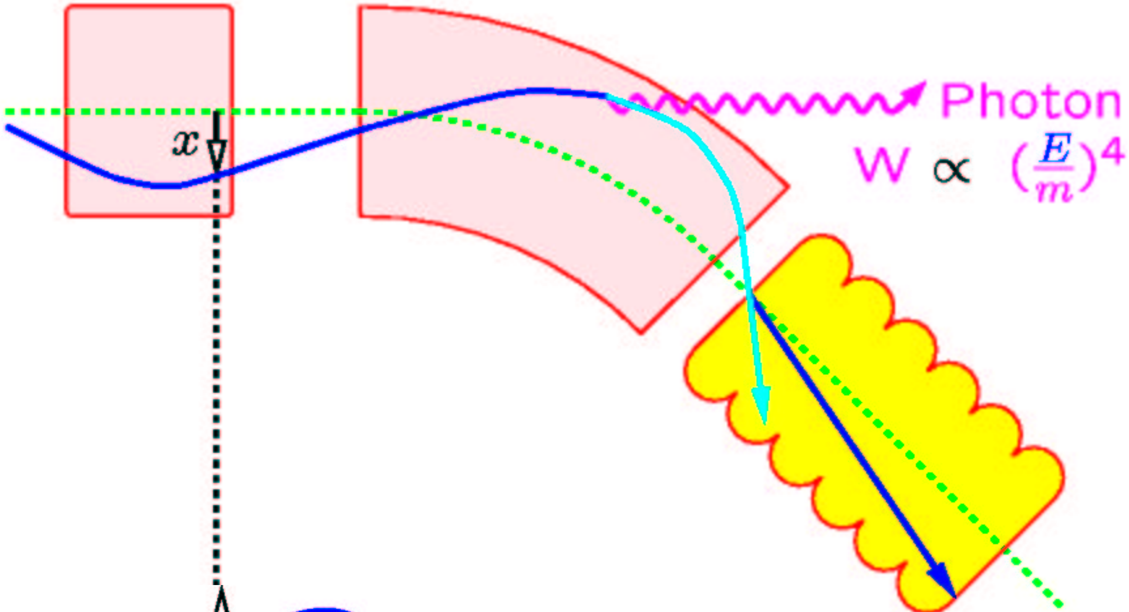
- A laser shines on a high voltage cathode, which emits photo electrons.
- These are accelerated either through an aperture in an anode (DC source), or in an RF field (RF photo-cathode source).
- With GaAs as cathode and with a polarized laser, polarized electrons are produced.
- Bunches can be as short as a few ps.
- Peak currents of a few 100A can be achieved.



Positron Source

- Electrons are accelerated to about 200MeV in a linac and hit a tungsten target.
- Pair production leads to e+/e- pairs.
- A following linac has the correct phase to accelerate e+ and decelerate e-.
- Due to multiple collisions in the target, the energy spread is up to 30MeV and
- The beam is very wide. A following damping ring is needed to produce narrow beams.

Creation of beam properties in a ring

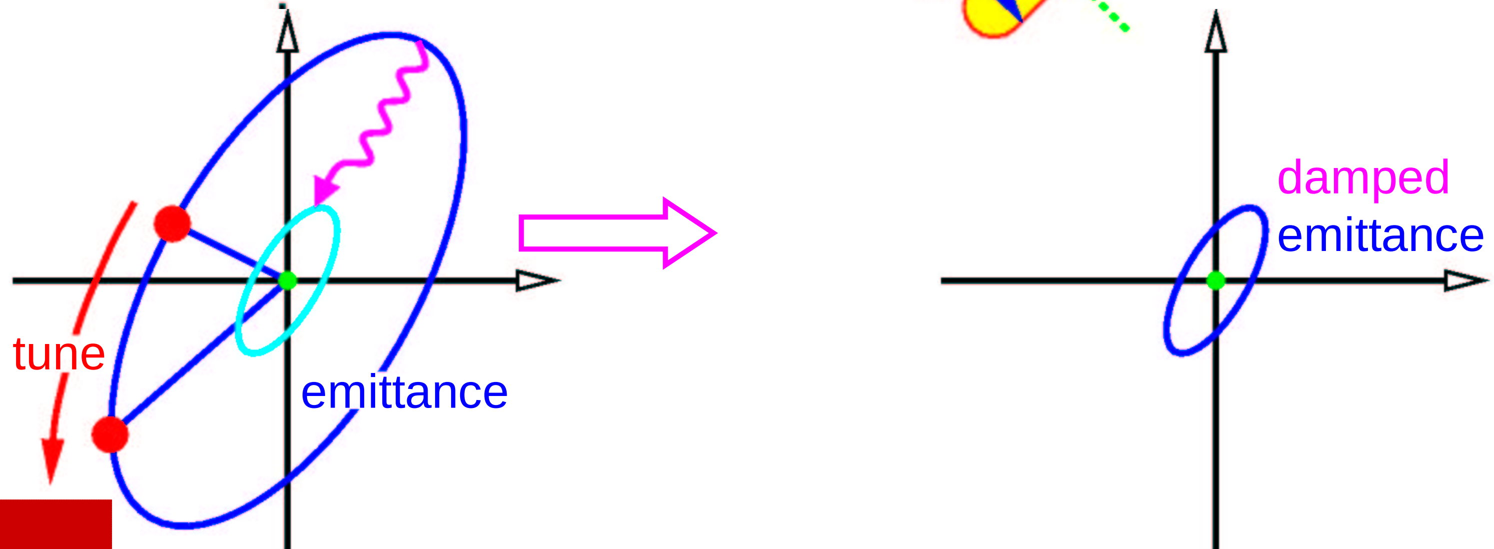


1) Energy emission:

Damping of the beam size

2) Stochastic emission:

Excitation of the beam size



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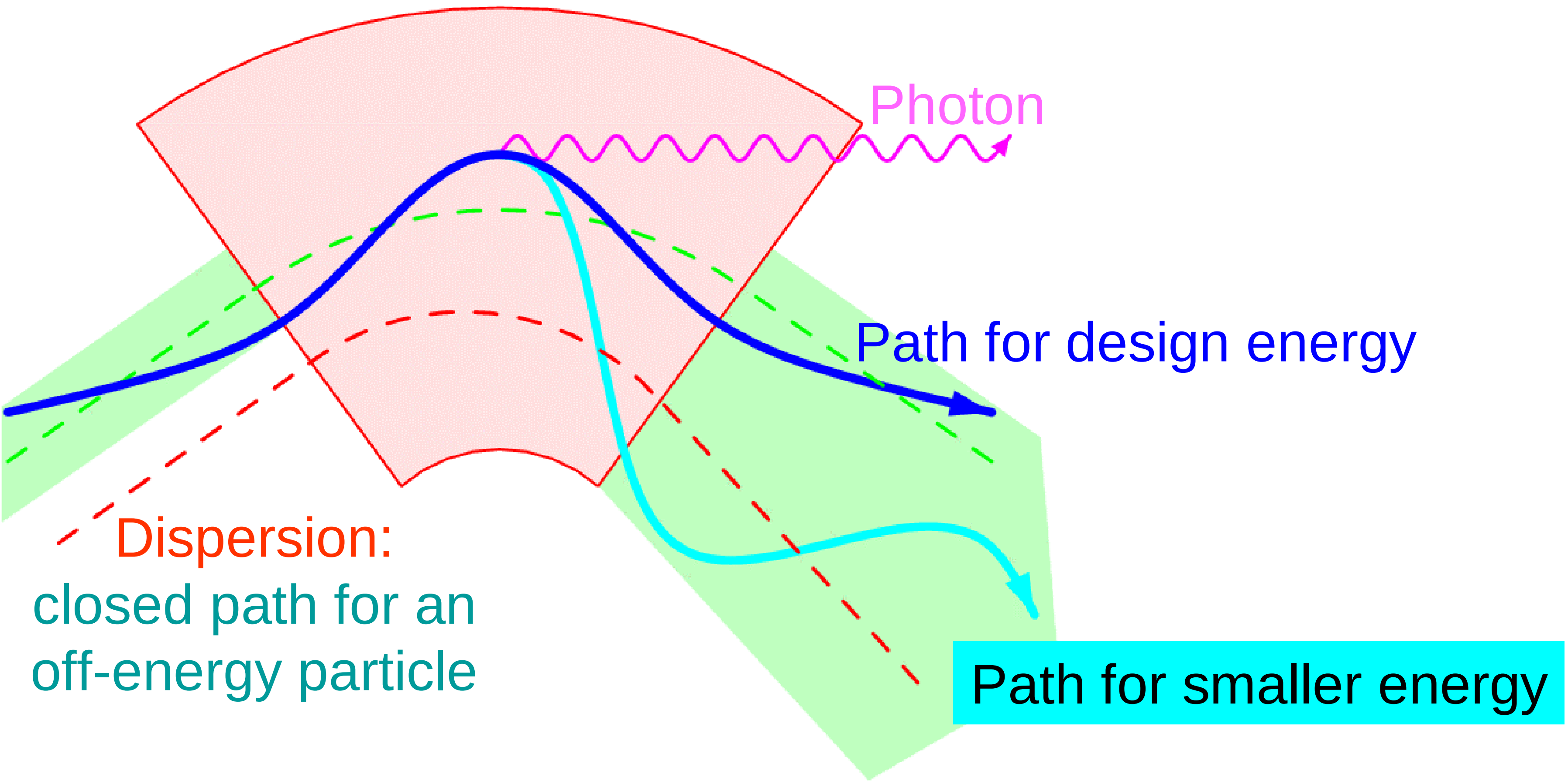
Generation of the emittance

Smaller dispersion → Smaller emittance



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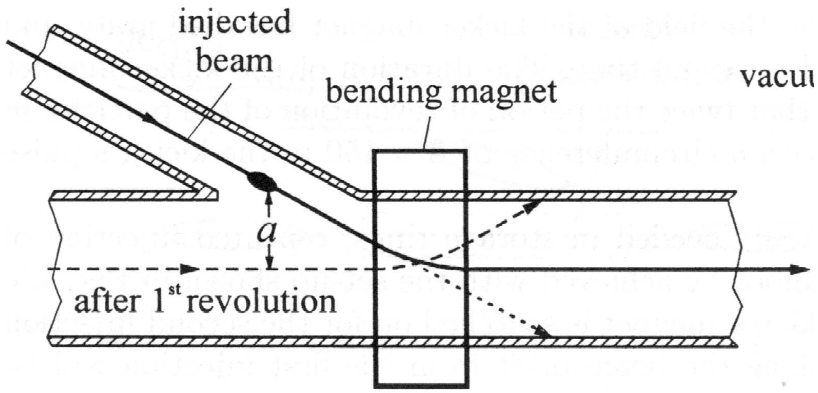
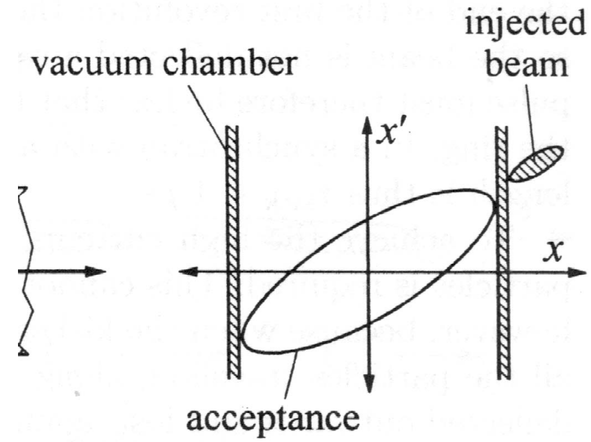
Injection and Extraction

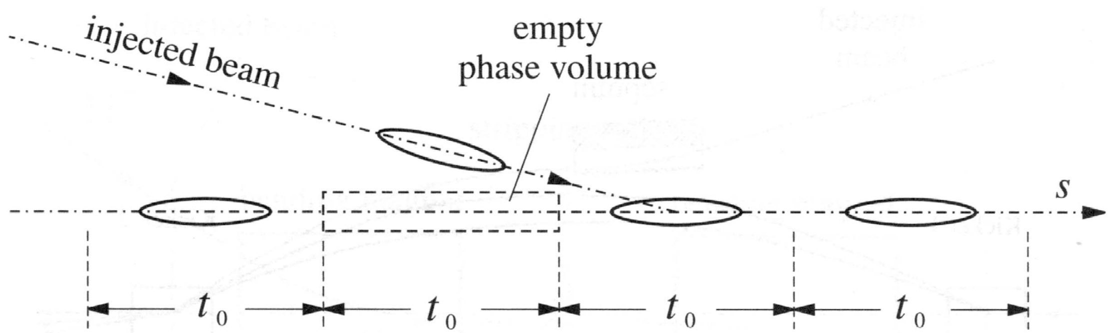




$$J_{inj} = \gamma a^2 + 2\alpha a a' + \beta a'^2 > \text{Aperture}$$

A fast kicker magnet is needed to bring a injected bunch onto the closed orbit.

- In order not to disturb the second turn, the duration of the kick must be less than 2 circulation times (1µs for a 150m ring).
- If the kicker magnet has fast enough rise time, one can inject many bunches .



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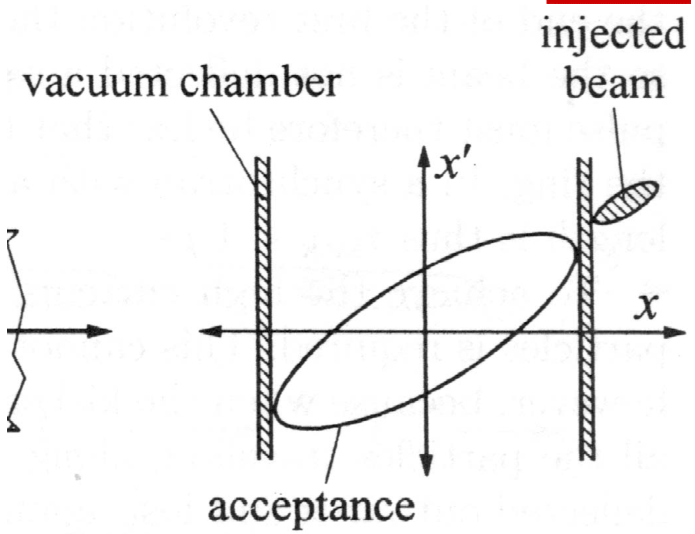
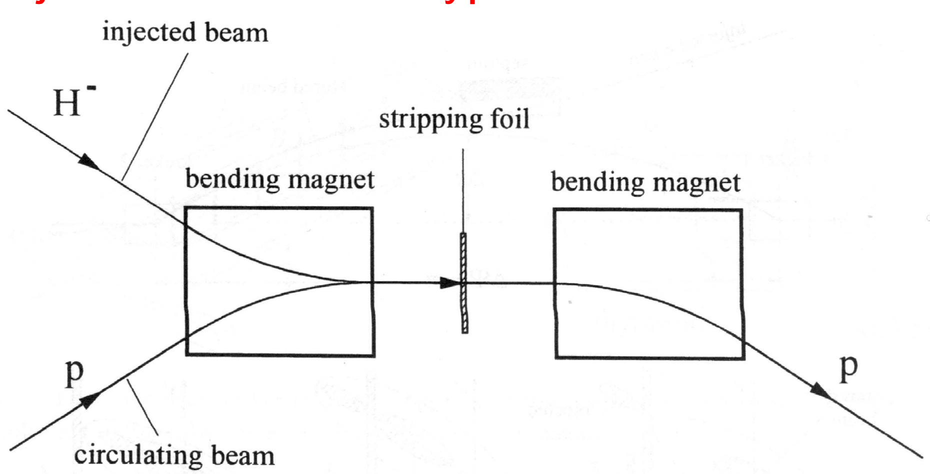
Injection and Lioville's Theorem

It is not possible to inject particles into an already occupied volume of phase space without losing the particles already present.

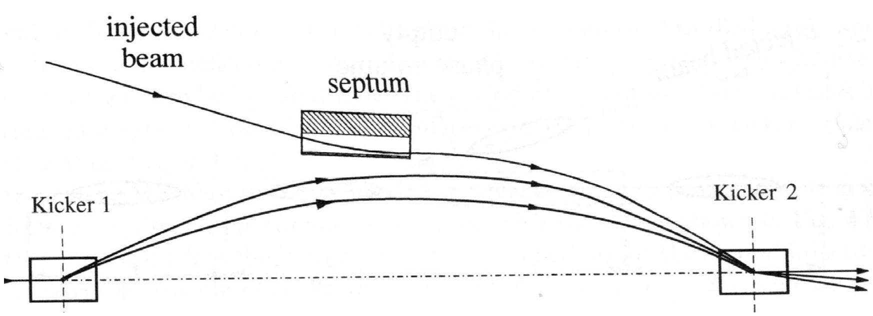


For Hamiltonian motion, two bunches of identical particles cannot merge in phase space, since the phase space density is conserved.

Injection of different types of bunches:



Phase space painting by a variable injection bump or energy:



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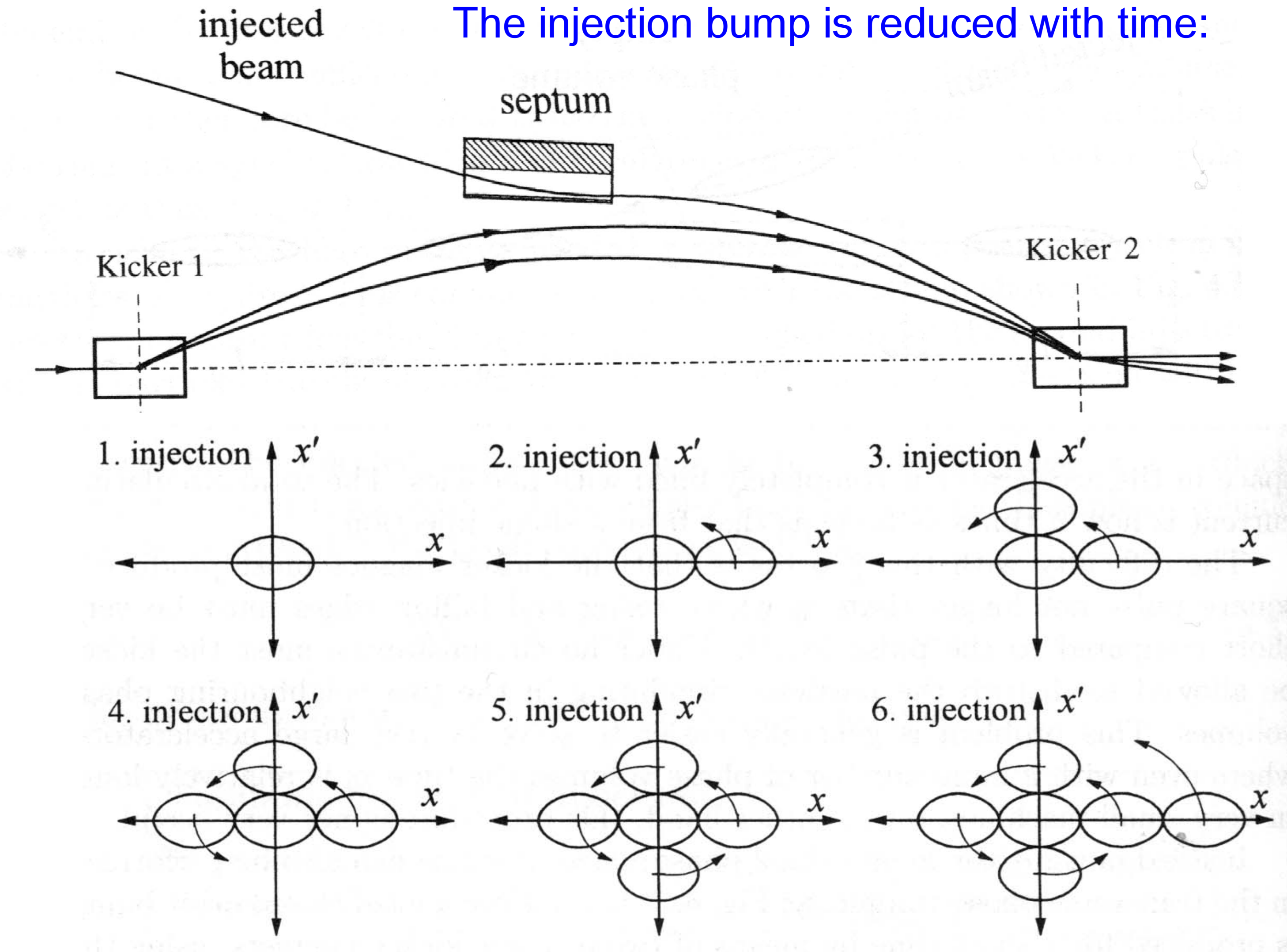
Phase Space Painting

The injection bump is reduced with time:

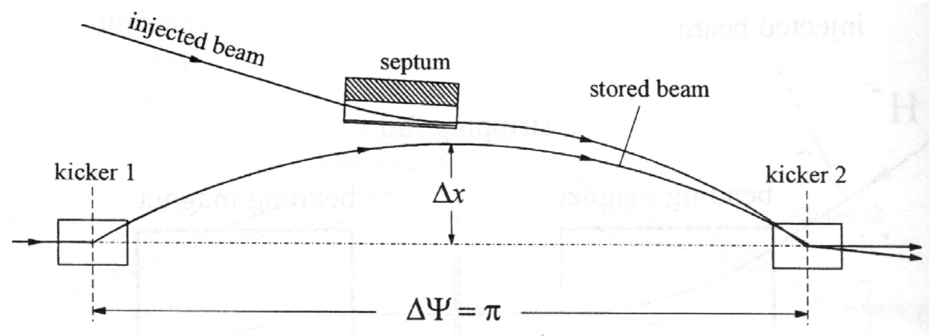


Each new injection fills an empty phase space area.

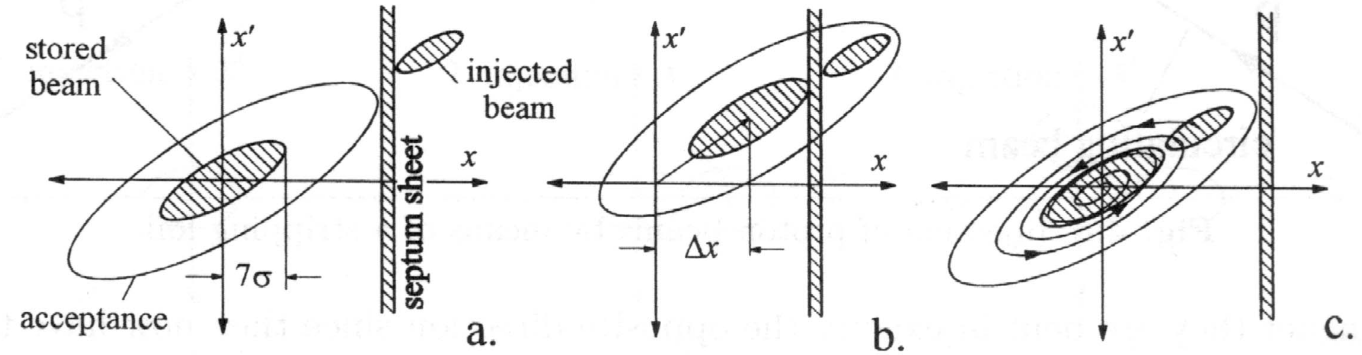
Injection for Electrons

Damping leads to non-Hamiltonian motion, so that Lioville's theorem does not told.

An injection bump brings the closed orbit, and possibly an existing beam, close
To the septum magnet, so that the new bunch is injected next to the existing beam.



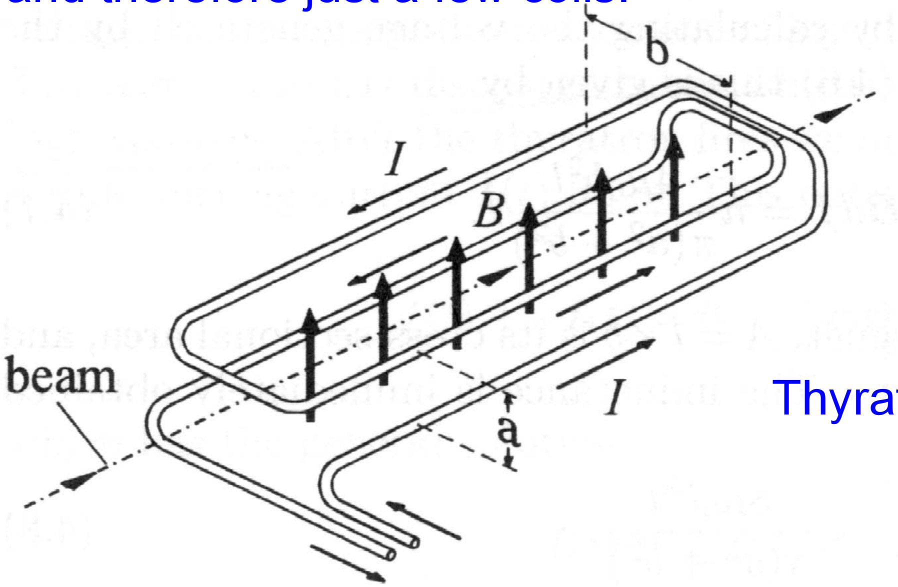
The injection oscillations of the new bunch damp in a few 100 turns
due to the emission of synchrotron radiation. There should be no island to damp to!



7s aperture is
Needed to not
loose stored
electrons.

Kicker Magnets

For a fast field change in a few ms, one need low inductance and therefore just a few coils.



Thyratrons are use as high current fast switch.

$$U = n\int_{\text{Area}} \dot{\vec{B}}\cdot d\vec{a} = n\frac{4\mu_0 b^2 l}{\pi(a^2+b^2)}\dot{I}$$

$$r = \tfrac{1}{2}\sqrt{a^2+b^2} \qquad \vec{B}^{(i)} = \frac{\mu_0 I}{2\pi r}\begin{pmatrix}\pm\frac{a}{r}\\ \frac{b}{r}\end{pmatrix}$$

$$B_y = \frac{\mu_0 b}{\pi(a^2+b^2)}I$$

$$L = \frac{U}{\dot{I}} = n\frac{4\mu_0 b^2 l}{\pi(a^2+b^2)}$$

(neglecting fringe fields and eddy-current shielding by the vacuum pipe.)

Kicker example

$$a = 0.04\text{m}$$

$$b = 0.08\text{m}$$

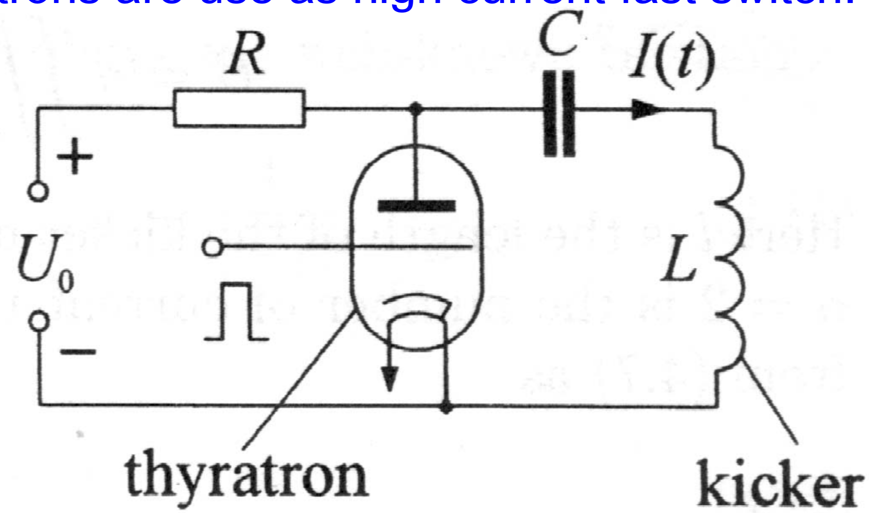
$$l = 1.0\text{m}$$

$$E = 5\text{GeV}$$

What current and what voltage is needed to produce a kick angle of

$\varphi = 3\text{mrad}$ in $\tau_{kick} = 1\mu s$

Thyratrons are use as high current fast switch.

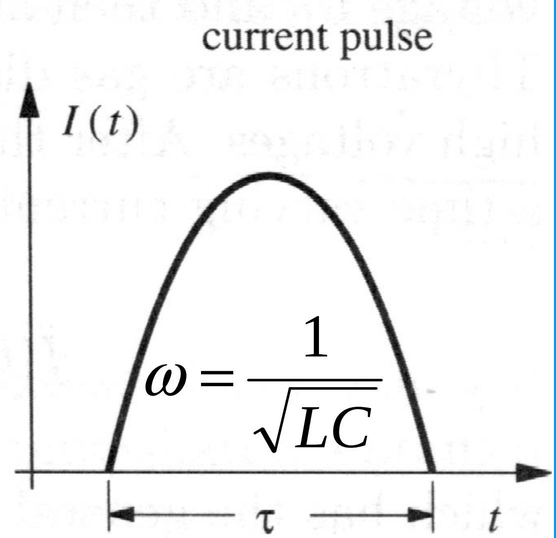


$$B_y = \frac{\mu_0 b}{\pi(a^2+b^2)} I = \frac{p}{e}\frac{\varphi}{l} \Rightarrow I = 3127\text{A}$$

$$L = \frac{U}{\dot I} = n\frac{4\mu_0 b^2 l}{\pi(a^2+b^2)} = 2.56\mu H$$

$$C = \left(\frac{\tau_{kick}}{\pi}\right)^2 \frac{1}{L} = 39.6nF$$

$$\hat U = L\hat{\dot I} = \omega L \hat I = 25.1\text{kV}$$



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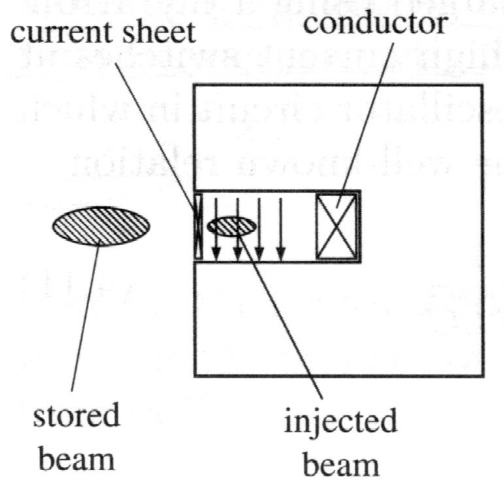
Septum magnets

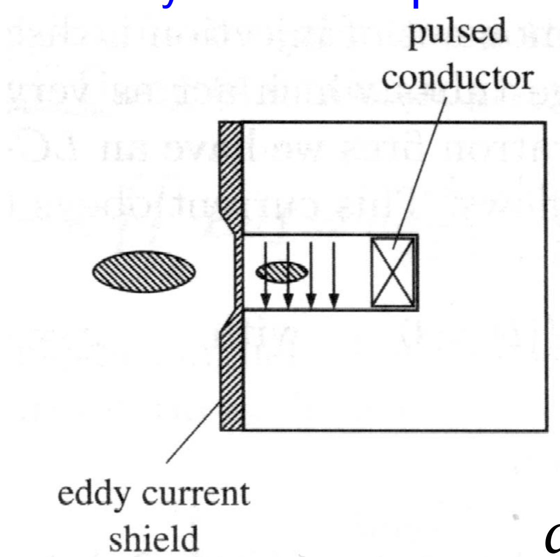
Current sheet septum magnet



The sheet has to carry the same current as the conductor. Its width is only a few mm.

For a few μs pulses, cooling is usually not required.

Eddy current septum magnet

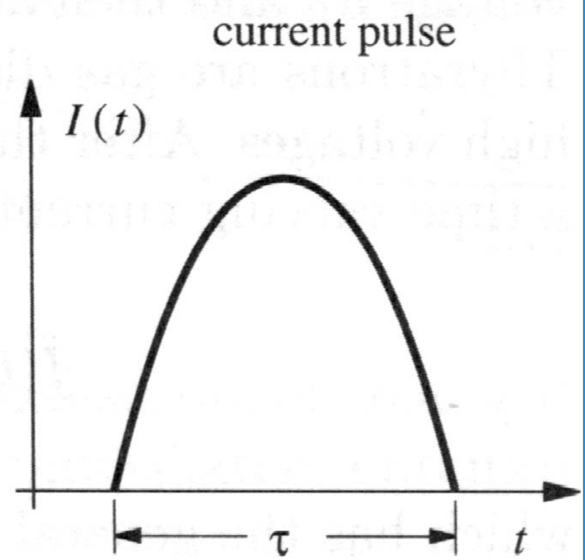


Eddy currents shield the stored Beam from the changing field.

It has to be significantly wider than the skin depth:

$$d_s = \sqrt{\frac{2}{\omega\sigma\mu_r\mu_0}}$$

$$d_s^{(Cu)} = 0.66\text{mm} \ \text{ for } \ 50\mu s$$



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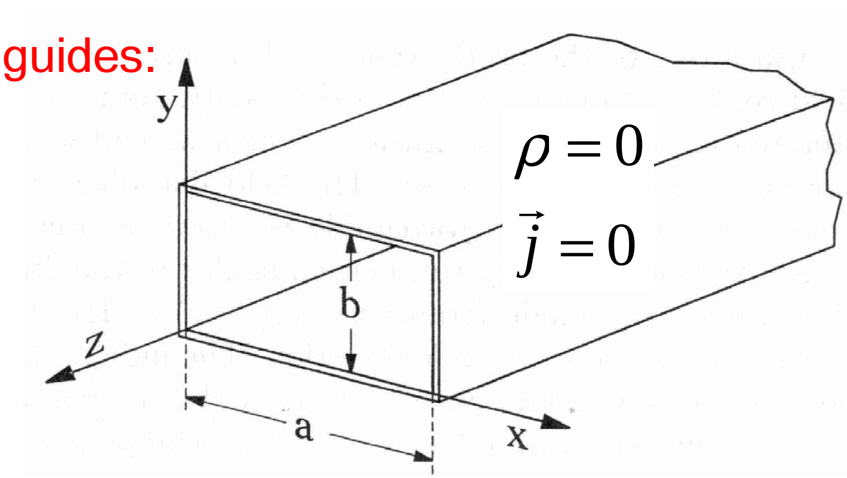
RF in Accelerators

$$\left.\begin{array}{l}\vec{\nabla}\times\vec{E}=-\partial_t\vec{B}\\ \vec{\nabla}\times\vec{B}=\frac{1}{c^2}\partial_t\vec{E}+\mu_0\vec{j}\end{array}\right\}\begin{array}{l}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=-\frac{1}{c^2}\partial_t^2\vec{E}-\mu_0\partial_t\vec{j}\\ \vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=-\frac{1}{c^2}\partial_t^2\vec{B}-\mu_0\vec{\nabla}\times\vec{j}\end{array}$$

Wave guides:



$$\rho=0$$
$$\vec{j}=0$$

Wave equation for all components

$$\vec{\nabla}^2\vec{E}=\frac{1}{c^2}\partial_t^2\vec{E}$$
$$\vec{\nabla}^2\vec{B}=\frac{1}{c^2}\partial_t^2\vec{B}$$

$$\left.\begin{array}{l}\vec{\nabla}\times\vec{E}=-\partial_t\vec{B}\\ \vec{\nabla}\times\vec{B}=\frac{1}{c^2}\partial_t\vec{E}\end{array}\right\}\begin{array}{l}\vec{\nabla}_\perp\times\vec{E}_\perp=-\partial_t\vec{B}_z\\ \vec{\nabla}_\perp\times\vec{B}_\perp=\frac{1}{c^2}\partial_t\vec{E}_z\end{array}$$

$$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{E}=0\\ \vec{\nabla}\cdot\vec{B}=0\end{array}\right\}\begin{array}{l}\vec{\nabla}_\perp\cdot\vec{E}_\perp+\partial_z E_z=0\\ \vec{\nabla}_\perp\cdot\vec{B}_\perp+\partial_z B_z=0\end{array}$$

Search for simple modes:

Transverse electric and magnetic (TEM) waves cannot exists, since:

$$E_z=0 \text{ and } B_z=0 \;\Rightarrow\; \vec{E}_\perp=0 \text{ and } \vec{B}_\perp=0$$

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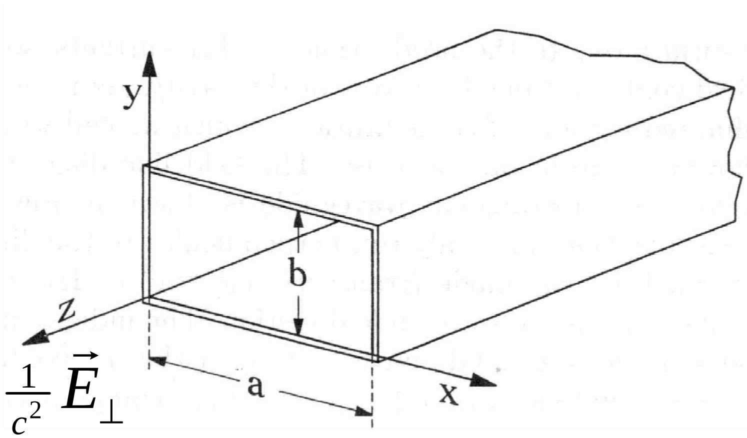
TE and TM Modes

Fourier expansion of the z-dependence: $\vec{E}(x,y,z,t)=\int \vec{E}_{k_z\omega}(x,y)e^{ik_z z-i\omega t}dk_z\,d\omega$

$$\vec{\nabla}^2\vec{E}=\tfrac{1}{c^2}\partial_t^2\vec{E} \qquad \vec{\nabla}_\perp^2 E_z=-[(\tfrac{\omega}{c})^2-k_z^2]E_z$$
$$\Rightarrow$$
$$\vec{\nabla}^2\vec{B}=\tfrac{1}{c^2}\partial_t^2\vec{B} \qquad \vec{\nabla}_\perp^2 B_z=-[(\tfrac{\omega}{c})^2-k_z^2]B_z$$

Eigenvalue equation with boundary conditions:



Walls:

$$\vec{E}_{//}=0 \quad \vec{B}_r=0$$

$$E_z=0 \quad \partial_r B_z=0$$

$$\vec{\nabla}\times\vec{B}=\tfrac{1}{c^2}\partial_t\vec{E}$$

$$\vec{\nabla}_\perp\times B_z+ik_z\vec{e}_z\times\vec{B}_\perp=-i\omega\tfrac{1}{c^2}\vec{E}_\perp$$

$$\vec{\nabla}_r\times B_z+ik_z\vec{e}_z\times\vec{B}_r=-i\omega\tfrac{1}{c^2}\vec{E}_\varphi\Rightarrow\partial_r B_z=0$$

Solutions for E or B only exist for a discrete set of eigenvalues: $(\tfrac{\omega}{c})^2-k_z^2=k_n^{(E)^2}$

$$(\tfrac{\omega}{c})^2-k_z^2=k_n^{(B)^2}$$

Due to different boundary conditions, E_z and B_z cannot simultaneously be nonzero.

TE modes have $E_z=0$ TM modes have $B_z=0$

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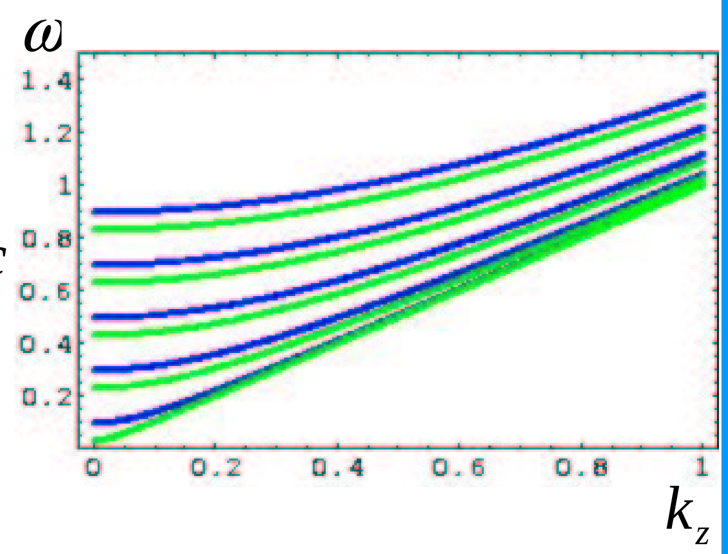
Dispersion relation

$$\omega(k_z) = c\sqrt{A_n{}^2 + k_z^2}$$



Phase velocity $v_{ph} = \omega / k_z = c\sqrt{1 + \left(\frac{A_n}{k_z}\right)^2} > c$

Group velocity $v_{gr} = d\omega / dk_z = c / \sqrt{1 + \left(\frac{A_n}{k_z}\right)^2} < c$

For each excitation frequency ω one obtains a propagation in the wave guide of

$$e^{ik_z z}\ , \quad k_z = \sqrt{(\tfrac{\omega}{c})^2 - A_n^2}$$

Transport for ω above the cutoff frequency $\omega > \omega_n = cA_n$

Damping for ω below the cutoff frequency $\omega < \omega_n = cA_n$

Rectangular Wave Guide

Boundary conditions:

$$E_z(\vec{x}_0)=0 \quad \vec{\nabla}_\perp^2 E_z = [k_z^2 - (\tfrac{\omega}{c})^2]E_z$$

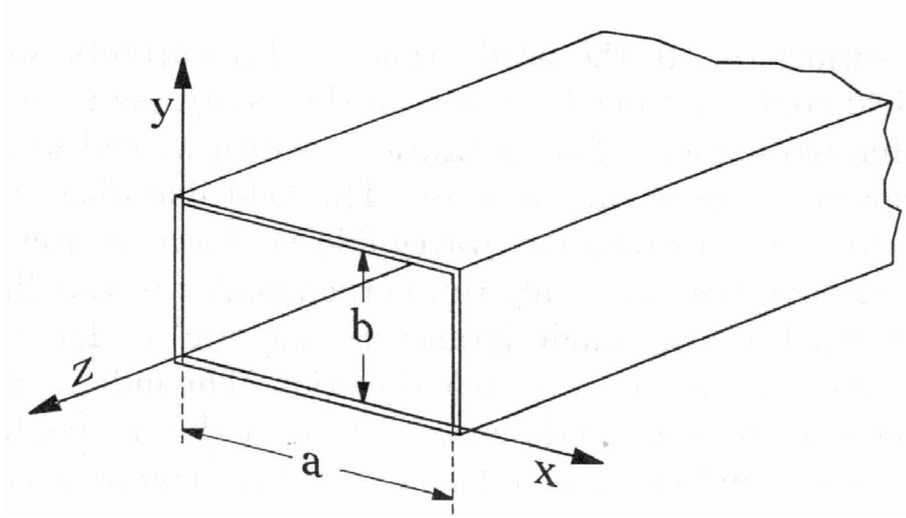
$$E_z(\vec{x}) = E_z \sin(\tfrac{n\pi}{a}x)\sin(\tfrac{m\pi}{b}y)$$

$$(\tfrac{\omega}{c})^2 - k_z^2 = k_{nm}^{(B)2} = \left(\tfrac{n\pi}{a}\right)^2 + \left(\tfrac{m\pi}{b}\right)^2$$

$$\partial_r B_z(\vec{x}_0)=0 \quad \vec{\nabla}_\perp^2 B_z = [k_z^2 - (\tfrac{\omega}{c})^2]B_z$$

$$B_z(\vec{x}) = B_z \cos(\tfrac{n\pi}{a}x)\cos(\tfrac{m\pi}{b}y)$$

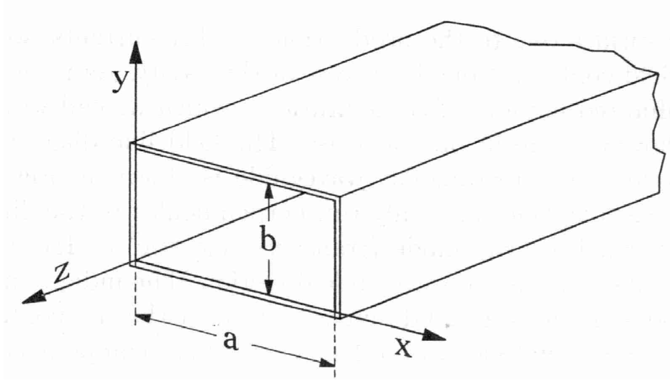
$$(\tfrac{\omega}{c})^2 - k_z^2 = k_{nm}^{(E)2} = \left(\tfrac{n\pi}{a}\right)^2 + \left(\tfrac{m\pi}{b}\right)^2$$



TE and TM modes happen to have the same eigenvalues.

For simplicity one still looks at TE and TM modes separately.

Rectangular TE Modes



Boundary conditions: $E_z(\vec{x}) = 0$

$$\vec{E}_{//}(\vec{x}_0) = 0$$

$$E_x(\vec{x}) = [A\cos(\tfrac{n\pi}{a}x) + B\sin(\tfrac{n\pi}{a}x)]\sin\tfrac{m\pi}{b}y)$$

$$E_y(\vec{x}) = \sin(\tfrac{n\pi}{a}x)[C\cos(\tfrac{m\pi}{b}y) + D\sin(\tfrac{m\pi}{b}y)]$$

$$\vec{\nabla}_\perp \cdot \vec{E}_\perp = 0 \quad \Rightarrow \quad D = 0\,, \quad B = 0\,, \quad C = -A\tfrac{n}{a}\tfrac{b}{m}$$

$$\vec{\nabla}_\perp \times \vec{E}_\perp = i\omega B_z \cos(\tfrac{n\pi}{a}x)\cos(\tfrac{m\pi}{b}y) \quad \Rightarrow \quad A\tfrac{b}{m\pi}\underbrace{[(\tfrac{m\pi}{b})^2 + (\tfrac{n\pi}{a})^2]}_{k_{nm}^{(E)2}} = -i\omega B_z$$

$$\vec{B}_r(\vec{x}_0) = 0 \qquad B_x(\vec{x}) = \sin(\tfrac{n\pi}{a}x)[C'\cos(\tfrac{m\pi}{b}y) + D'\sin(\tfrac{m\pi}{b}y)]$$

$$B_y(\vec{x}) = [A'\cos(\tfrac{n\pi}{a}x) + B'\sin(\tfrac{n\pi}{a}x)]\sin(\tfrac{m\pi}{b}y)$$

$$\vec{\nabla}_\perp \times \vec{B}_\perp = 0 \quad \Rightarrow \quad D' = 0\,, \quad B' = 0\,, \quad C' = A'\tfrac{n}{a}\tfrac{b}{m}$$

$$\vec{\nabla}_\perp \cdot B_\perp = -ik_z B_z \cos(\tfrac{n\pi}{a}x)\cos(\tfrac{m\pi}{b}y) \quad \Rightarrow \quad A'\tfrac{b}{m\pi}k_{nm}^{(E)2} = -ik_z B_z$$

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Rectangular TE and TM Modes

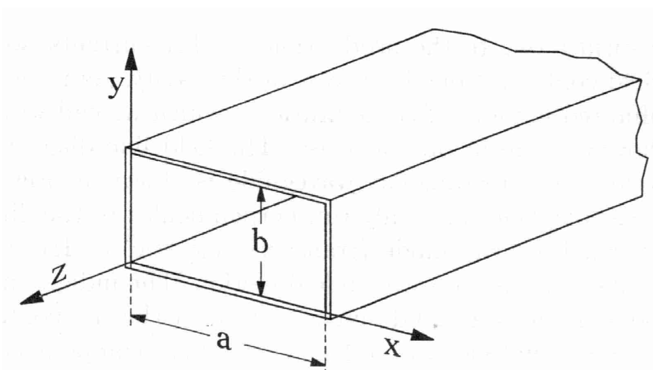
TE Modes

$$\vec{B}(\vec{x}) = B_z \begin{pmatrix} \frac{n\pi}{a}\frac{k_z}{k_{nm}^{(E)2}}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ \frac{m\pi}{b}\frac{k_z}{k_{nm}^{(E)2}}\cos(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ \cos(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\cos(k_z z-\omega t) \end{pmatrix}$$

$$\vec{E}(\vec{x}) = \frac{\omega}{k_{nm}^{(E)2}} B_z \begin{pmatrix} \frac{m\pi}{b}\cos(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ -\frac{n\pi}{a}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ 0 \end{pmatrix}$$

TM Modes: Exchange of E and B

Notation: TE_{nm} Mode

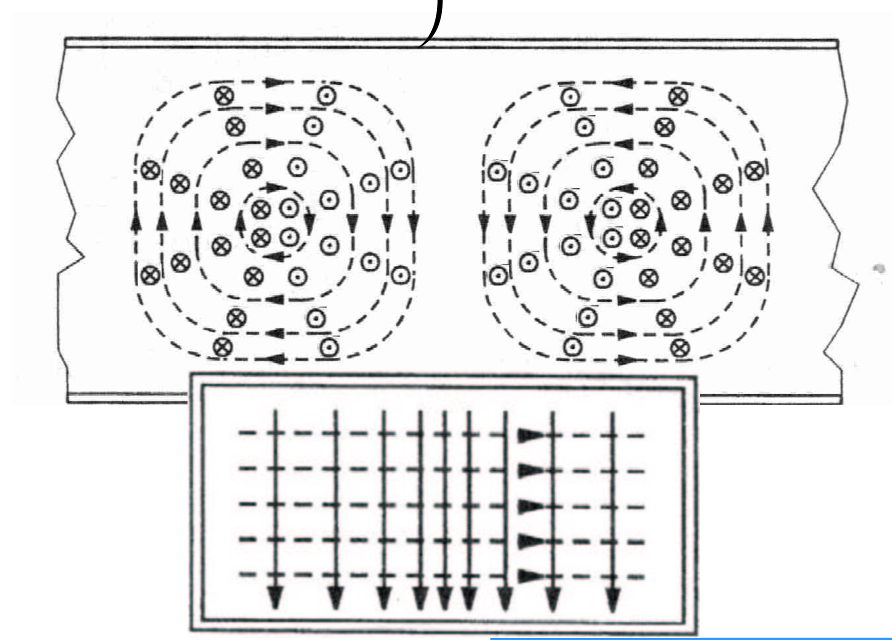


$\vec{E}$ ⟶

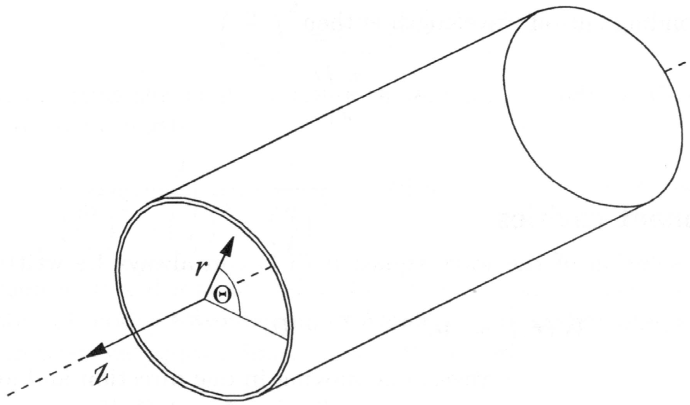
$\vec{B}$ - - ➝

$n=1$

$m=0$



Cylindrical Wave Guides



TM Modes:

$$E_z(\vec{x}_0) = 0 \qquad \vec{\nabla}_\perp^2 E_z = [k_z^2 - (\tfrac{\omega}{c})^2] E_z$$

$$(\partial_r^2 + \tfrac{1}{r}\partial_r + \tfrac{1}{r^2}\partial_\varphi^2) E_z = [k_z^2 - (\tfrac{\omega}{c})^2] E_z$$

$$(\xi^2\partial_\xi^2 + \xi\partial_\xi + \xi^2 - n^2) E_z = 0\,, \quad \xi = k_{nm}^{(E)}\, r$$

$$E_z(\vec{x}) = E_z J_n(k_{nm}^{(B)} r) e^{in\varphi}$$

$k_{nm}^{(B)}$ is the mth 0 of the nth Bessel function over r.

TE Modes:

$$\partial_r B_z(\vec{x}_0) = 0 \quad \vec{\nabla}_\perp^2 B_z = [k_z^2 - (\tfrac{\omega}{c})^2] B_z$$

$$B_z(\vec{x}) = B_z J_n(k_{nm}^{(E)} r) e^{in\varphi}$$

$k_{nm}^{(E)}$ is the mth extremeum of J_n over r.

Notation: TE_{nm} Mode

Fundamental Mode

11/25/03
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Mode for particle acceleration: TM_{01}

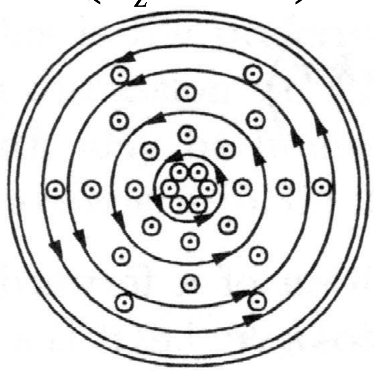
$$E_z(\vec{x}) = E_z J_0(\tfrac{r}{r_0})\cos(k_z z - \omega t)$$

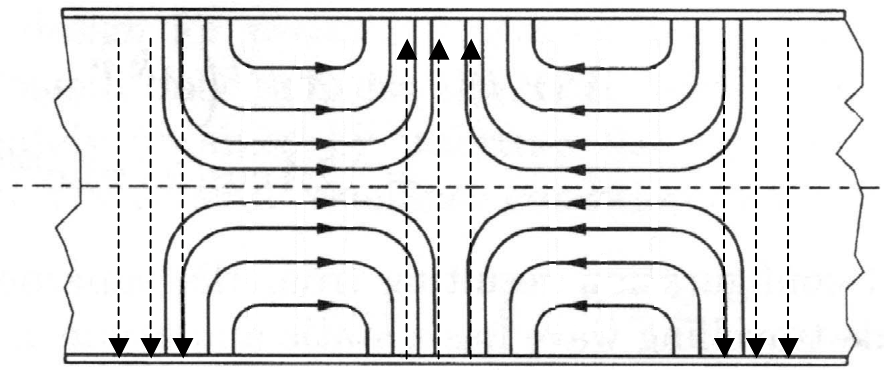
$$E_r(\vec{x}) = -E_z r_1 k_z J_0'(\tfrac{r}{r_1})\sin(k_z z - \omega t)$$

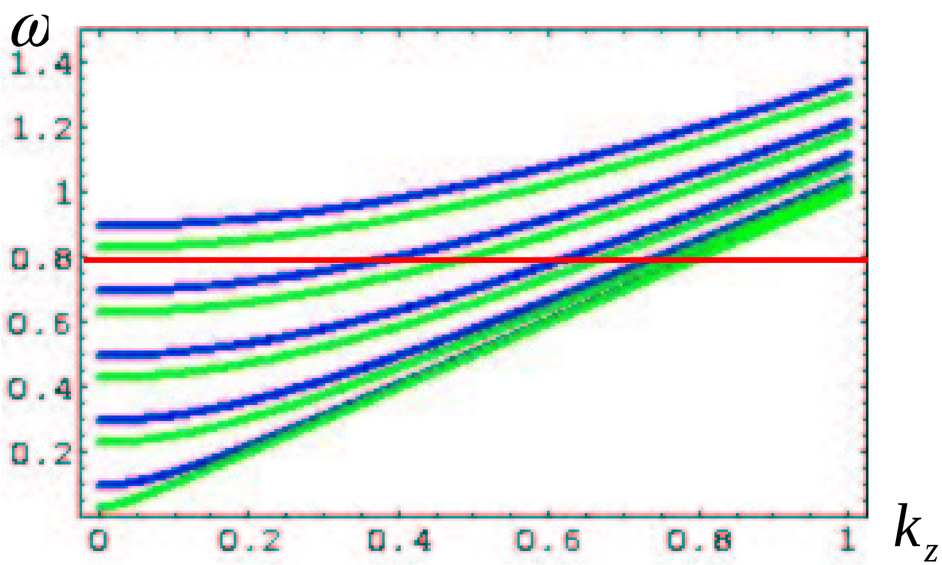
$$E_\varphi(\vec{x}) = 0$$

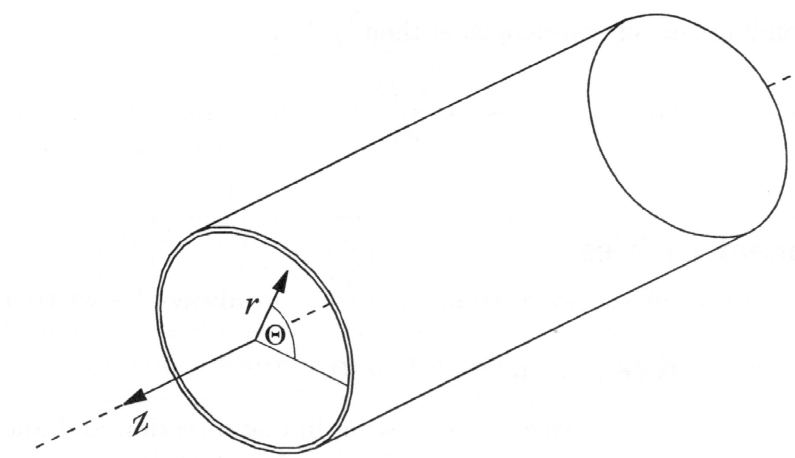
$$B_r(\vec{x}) = 0$$

$$B_\varphi(\vec{x}) = -E_z r_1 \tfrac{\omega}{c^2} J_0'(\tfrac{r}{r_1})\sin(k_z z - \omega t)$$









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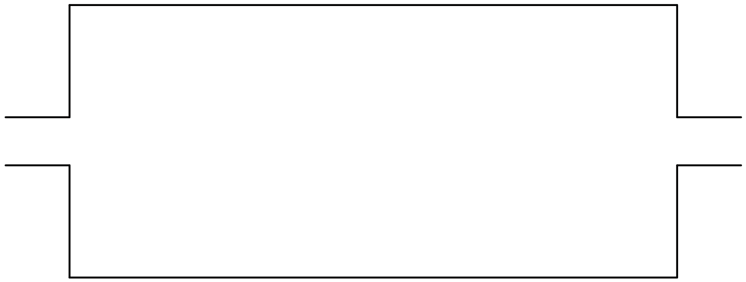
Resonant Cavities

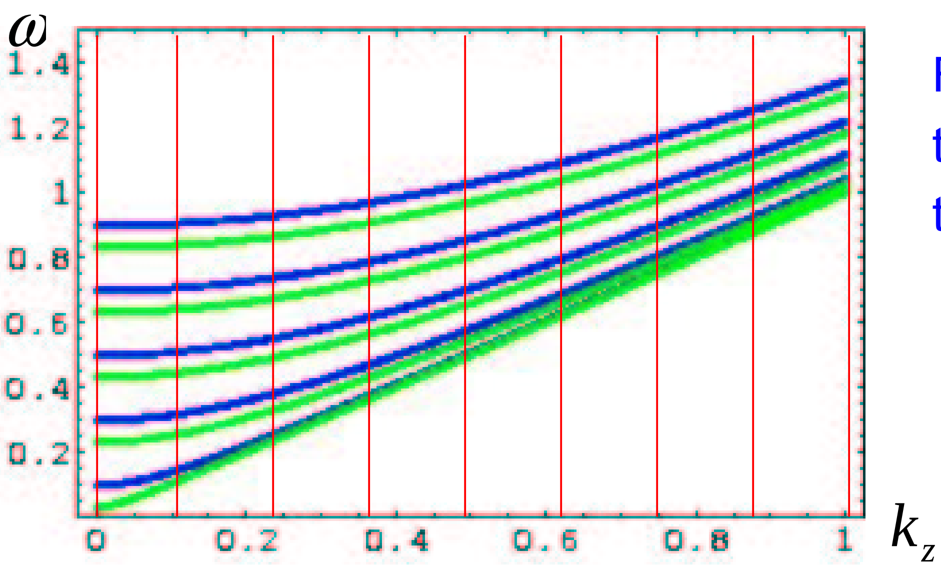


TE Modes: Standing waves with nodes

$$B_z(\vec{x}) \propto \sin(k_z z)\sin(\omega t)\,, \quad k_z = \tfrac{l\pi}{L}$$
$$l > 0$$

TM Modes: Standing waves with nodes

$$E_z(\vec{x}) \propto \cos(k_z z)\cos(\omega t)\,, \quad k_z = \tfrac{l\pi}{L}$$
$$l \geq 0$$

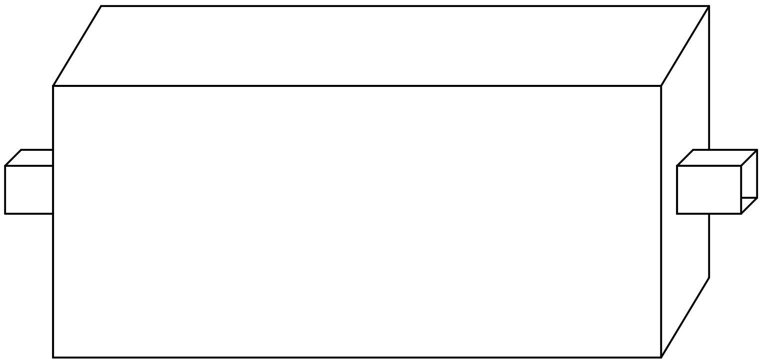


For each mode TE_{nm} or TM_{nm} there is a discrete set of frequencies that can be excited.

$$\omega_{nm}^{(E/B)} = c\sqrt{k_{nm}^{(E/B)2} + \left(\tfrac{l\pi}{L}\right)^2}$$

Resonant Cavities Examples

Rectangular cavity:

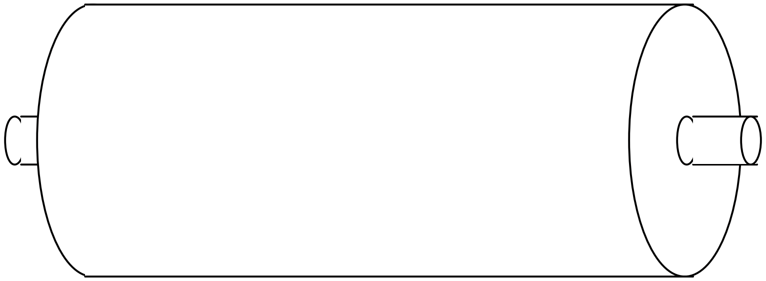


$$\omega_{nm}^{(E/B)} = c\sqrt{\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{l\pi}{L_z}\right)^2}$$

Fundamental acceleration mode: $\omega_{11}^{(B)} = c\frac{\pi}{L}\sqrt{2}$

$$L_x = L_y = 22cm \quad \Rightarrow \quad f_{110}^{(B)} = 1.0\text{GHz}$$

Pill Box cavity:



$$\omega_{nm}^{(E/B)} = c\sqrt{k_{nm}^{(E/B)2} + \left(\frac{l\pi}{L}\right)^2}$$

$k_{nm}^{(B)} r$ is the m[th] 0 of the n[th] Bessel function.

$k_{nm}^{(E)} r$ is the m[th] extremeum of J_n

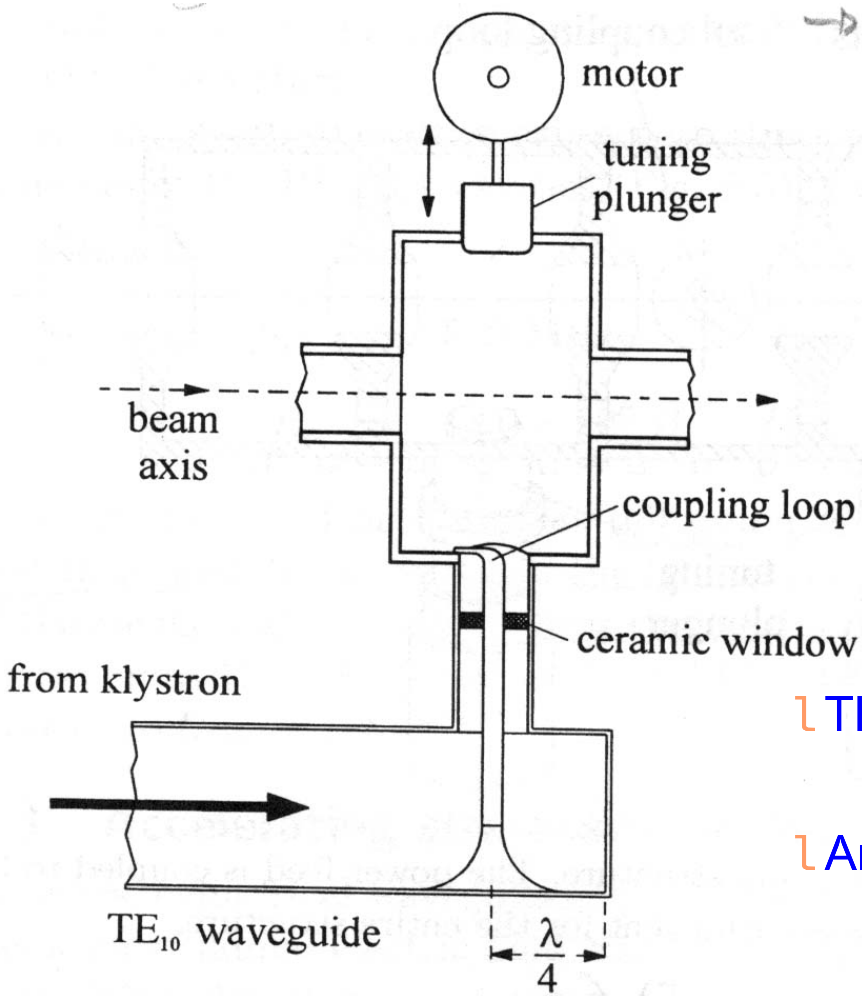
Fundamental acceleration mode: $\omega_{01}^{(E)} = c\frac{2.4}{r}$

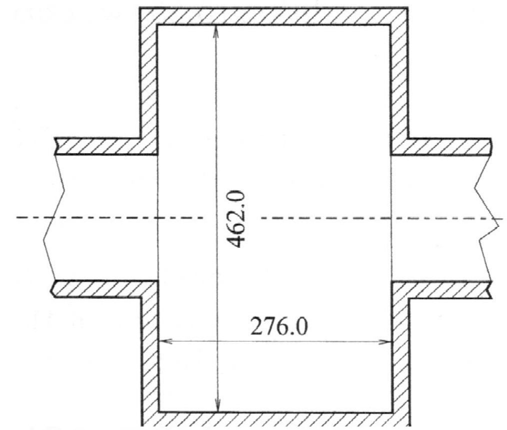
$$r = 11cm \quad \Rightarrow \quad f_{010}^{(M)} = 1.0\text{GHz}$$

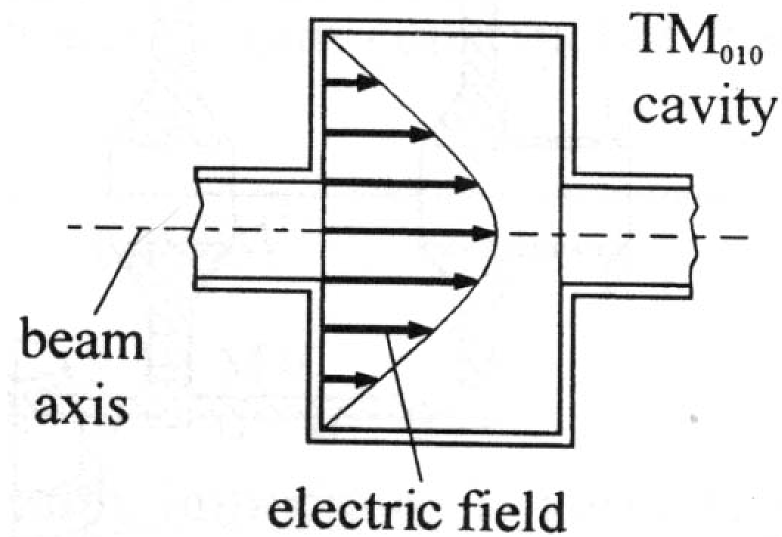
Cavity Operation

500MHz Cavity of DORIS:

$$r = 23.1cm \quad \Rightarrow \quad f_{010}^{(M)} = 0.4967\text{GHz}$$







- The frequency is increased and tuned by a tuning plunger.
- An inductive coupling loop excites the magnetic field at the equator of the cavity.

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RF systems for accelerators

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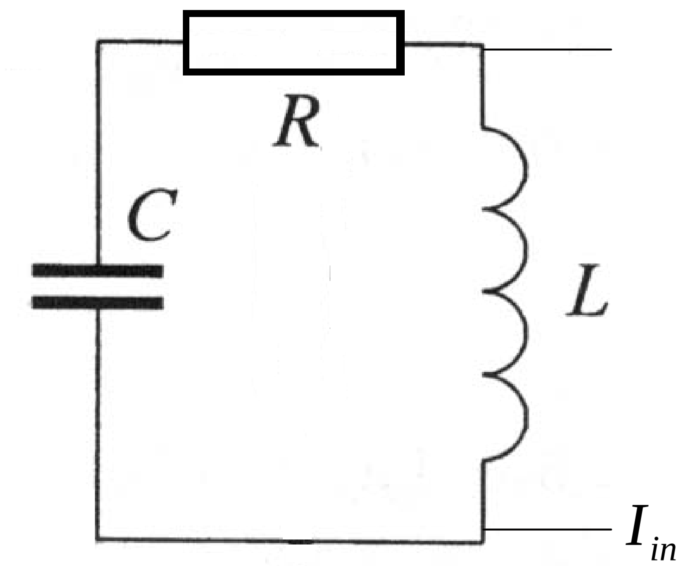

L and C: determined by the cavity geometry

R : determined by the surface resistance

$$U_C = \int \frac{I_C}{C} dt \to -i \frac{I_C}{C\omega}$$

$$L(\dot{I}_{in} - \dot{I}_C) = R I_C + \int \frac{I_C}{C} dt$$

$$I_C = \left(R - i \frac{1}{C\omega} + iL\omega \right)^{-1} iL\omega I_{in}$$

$$\hat{U}_C = \frac{1}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega \right)^2}} \frac{L}{C} \hat{I}_{in} \quad \to \quad \hat{U}_{Cres} = \frac{L}{RC} \hat{I}_{in} \,, \quad \omega_{res} = \frac{1}{\sqrt{LC}}$$

$$P_{RF} = \langle U_L I_{in} \rangle = \left\langle (R + \tfrac{1}{iC\omega}) R \tfrac{1}{iL\omega} I_C^2 \right\rangle = \left\langle (iC\omega R + 1) R \tfrac{C}{L} U_C^2 \right\rangle = \tfrac{1}{2} \tfrac{C}{L} R \sqrt{\tfrac{C}{L} R^2 + 1} \hat{U}_C^2$$

(An alternative circuit diagram leads to simplified formulas)

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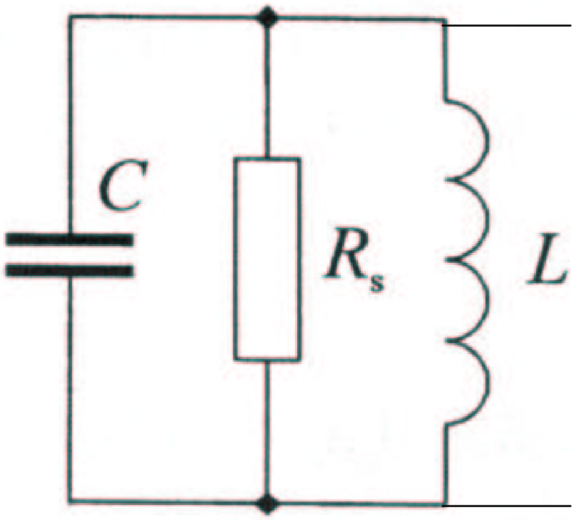
RF systems for accelerators



L and C: determined by the cavity geometry

R_s : shunt impedance, related to surface res. R

$$I_{in} = \left(\frac{1}{R_s} + iC\omega + \frac{1}{iL\omega} \right) U_C$$

$$\hat{U}_C = \frac{1}{\sqrt{\frac{1}{R_s^2} + \left(\frac{1}{L\omega} - C\omega \right)^2}} \hat{I}_{in} \rightarrow \hat{U}_{Cres} = R_s \hat{I}_{in}$$

$$P_{RF} = \langle U_L I_{in} \rangle = \tfrac{1}{2} \frac{1}{R_s} \hat{U}_C^2 \qquad \hat{U}_C = \sqrt{2 R_s P_{RF}}$$

Quallity factor: $$Q = 2\pi \frac{E}{\Delta E} = 2\pi \frac{\frac{1}{2} C U_C^2}{T P_{RF}} = \omega R_s C = R_s \sqrt{\tfrac{C}{L}}$$

Geometry factor: $$\frac{R_s}{Q} = \sqrt{\frac{L}{C}}$$

Superconducting Cavities

Q = 10^{10}

E = 20MV/m

A bell with this Q would ring for a year.

- Very low wall losses.
- Therefore continuous operation is possible.
- Energy recovery becomes possible.

Normal conducting cavities

- Significant wall losses.
- Cannot operate continuously with appreciable fields.
- Energy recovery was therefore not possible.

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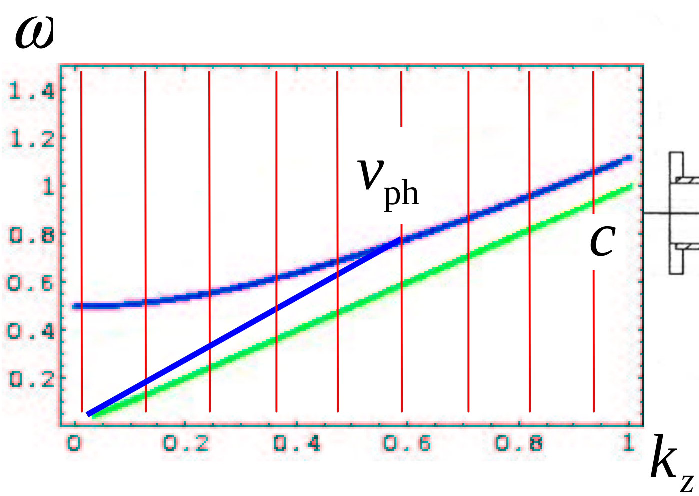
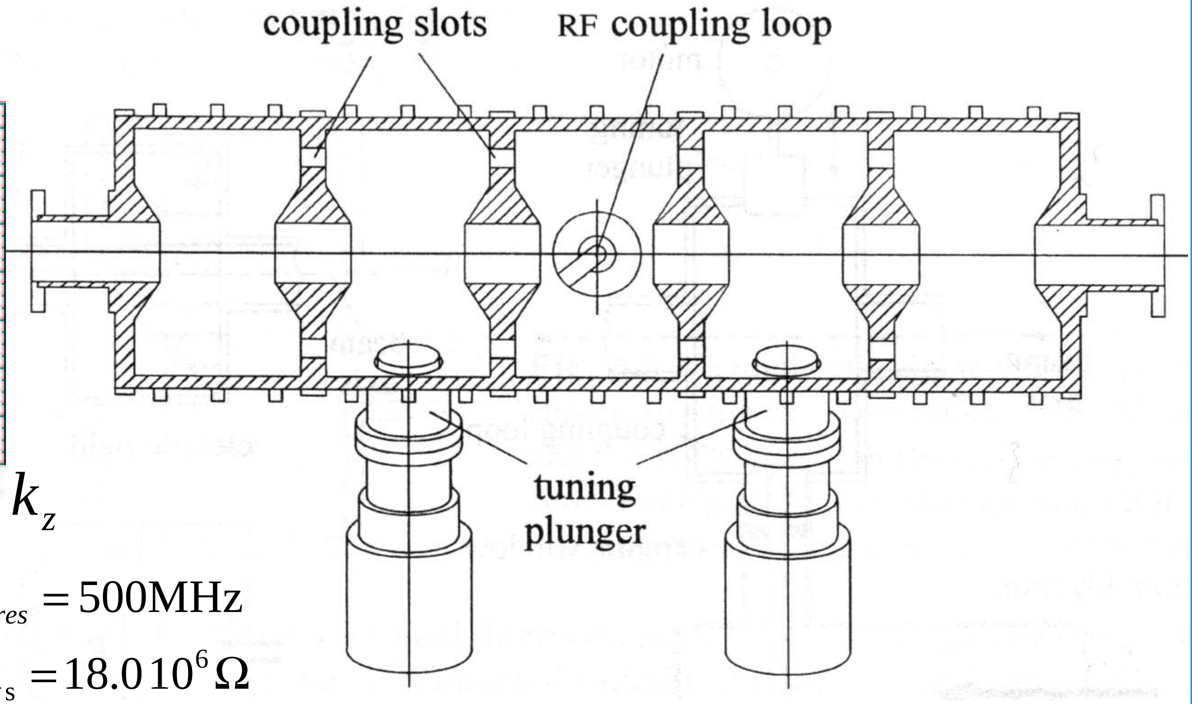
Multicell Cavities

The filed in many cells can be excited by a single power source and a single input coupler in order to have the voltage of several cavities available.





Example: PETRA Cavity

$f_{res} = 500\text{MHz}$

$R_s = 18.0\,10^6\,\Omega$

$125\text{kW} \rightarrow 2.12\text{MV}$

Without the walls: Long single cavity with too large wave velocity. $v_{ph} = \frac{\omega}{k}$

Thick walls: shield the particles from regions with decelerating phase.

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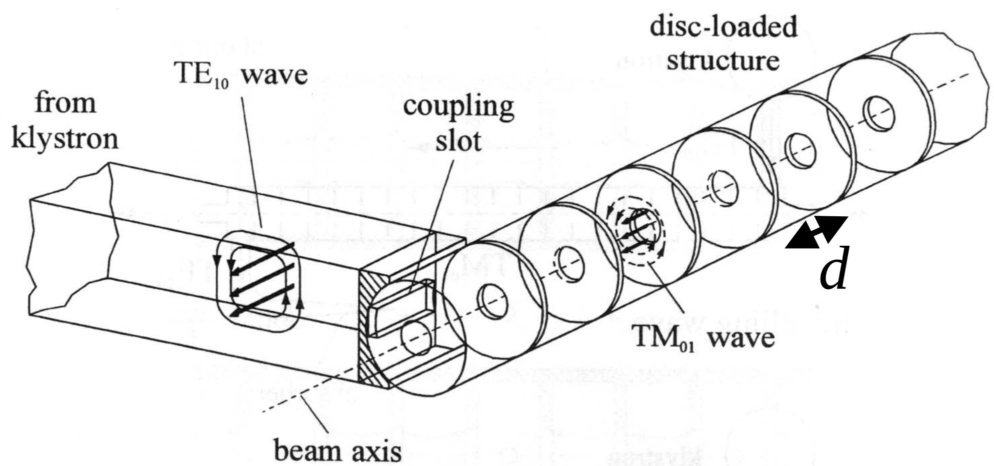
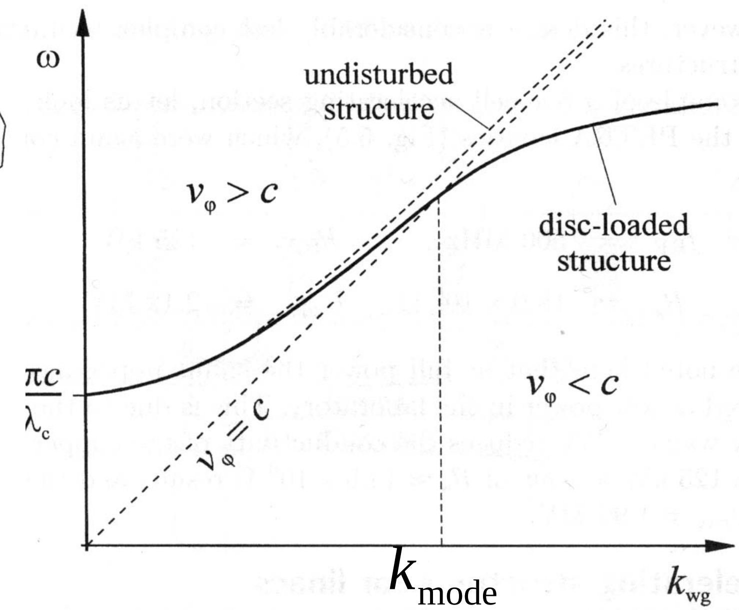
Disc Loaded Waveguides

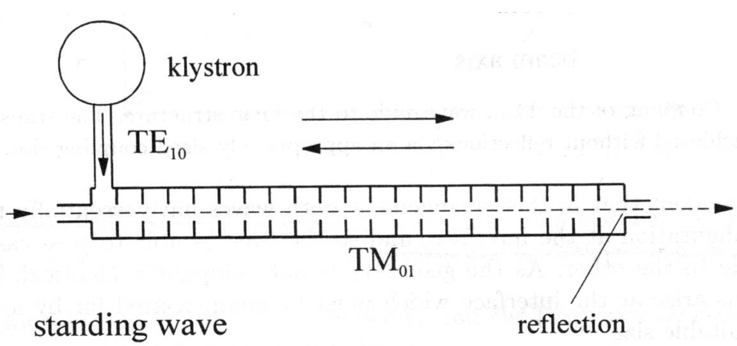
The iris size is chosen to let the phase velocity equal the particle velocity.



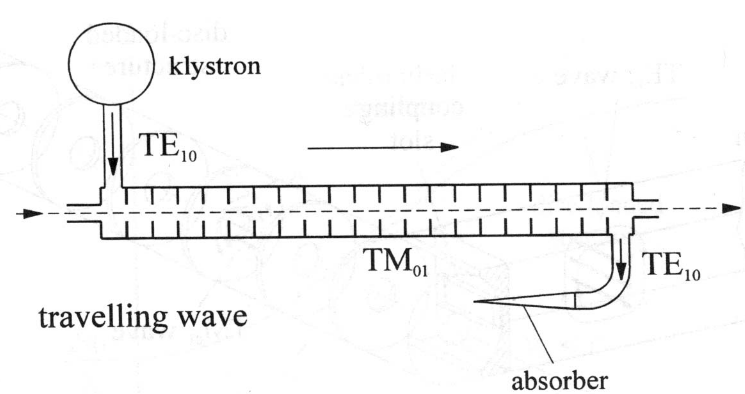


Loss free propergation: $k = \frac{2\pi}{nd}$

Standing wave cavity.



Traveling wave cavity (wave guide).



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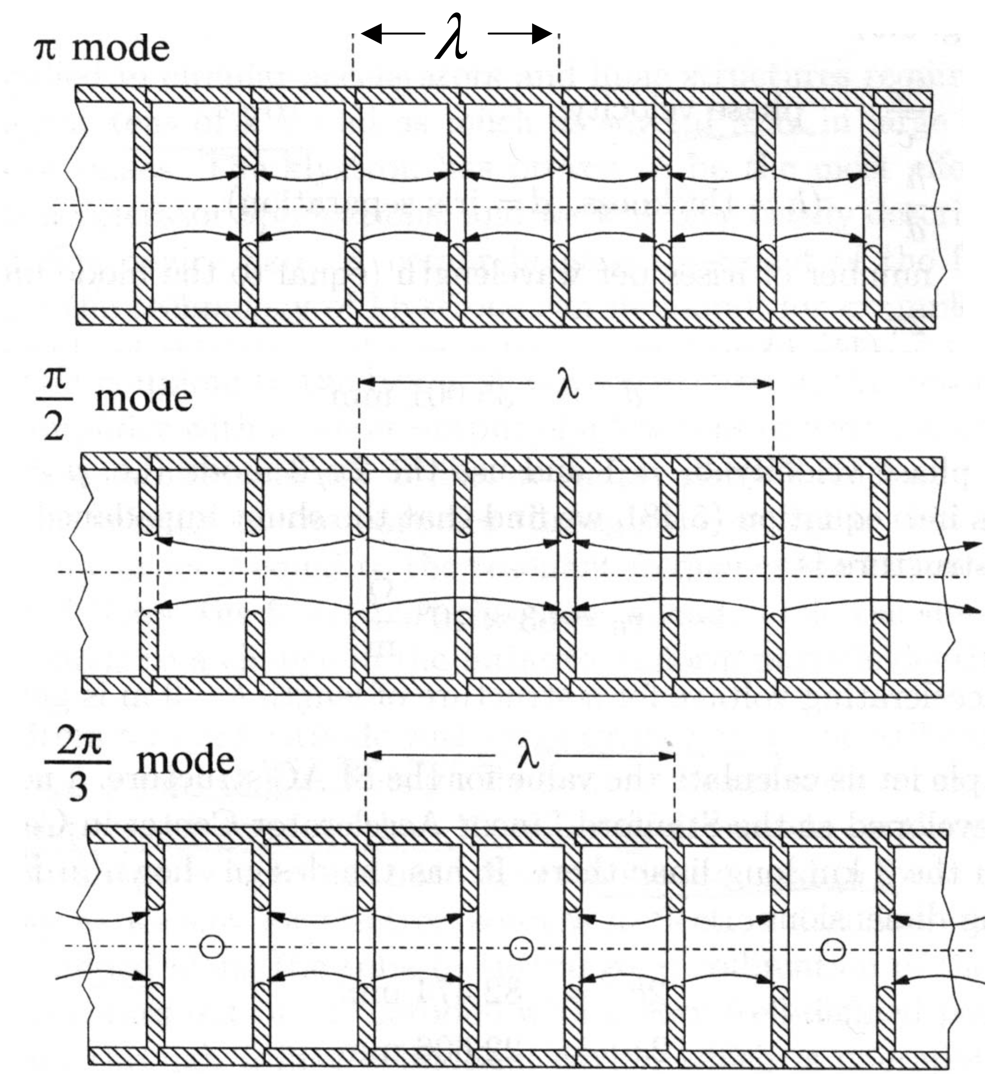
Modes in Waveguides

The iris size is chosen to let the phase velosity equal the particle velocity.



Long initial settling or filling time, not good for pulsed operation.

Small shunt impedance per length.

Common compromise.

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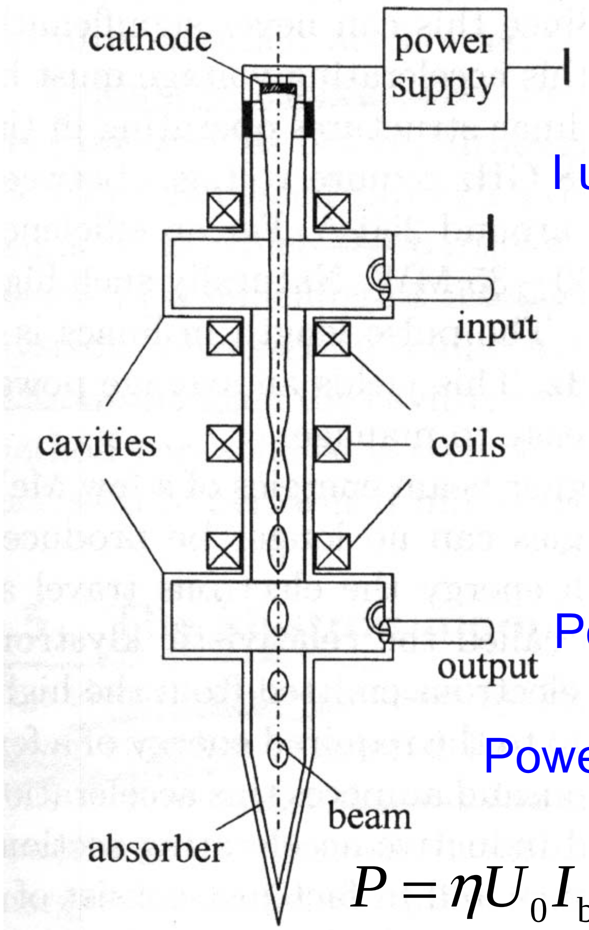
The Klystron as Power Source



I up to > 10A

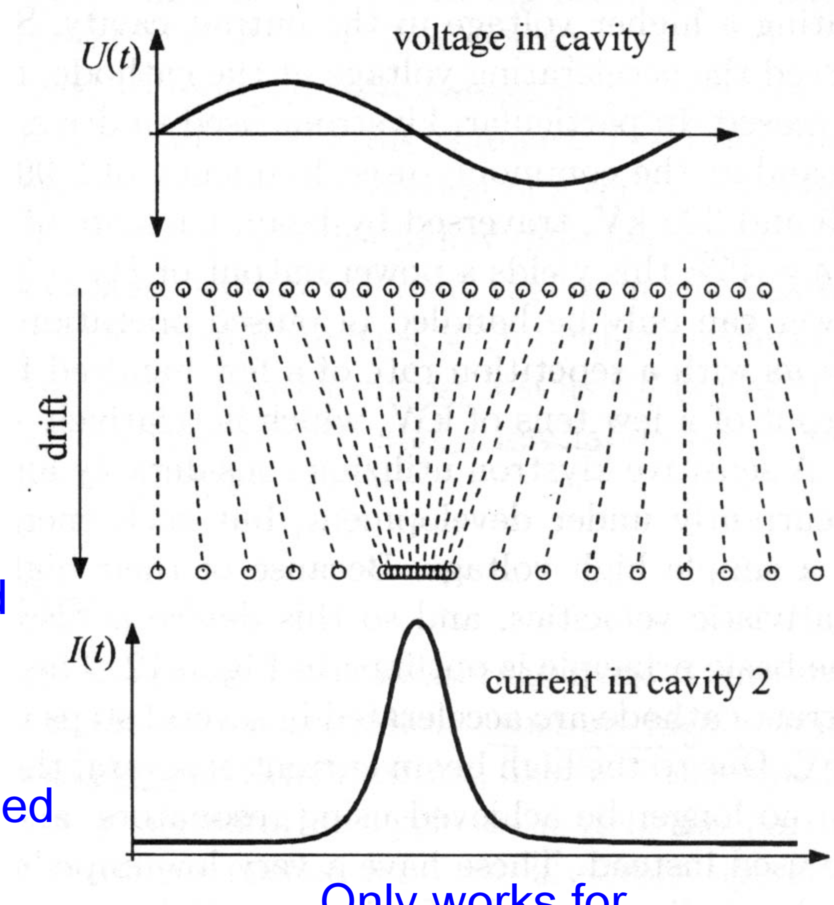
Power < 1.5MW

Power < 40MW pulsed

$$P = \eta U_0 I_{\text{beam}} \,, \quad \eta \le 65\%$$

- DC acceleration to several 10kV, 100kV pulsed
- Energy modulation with a cavity
- Time of flight density modulation
- Excitation of a cavity with output coupler

Time of flight bunching



Only works for

non-relativistic electrons

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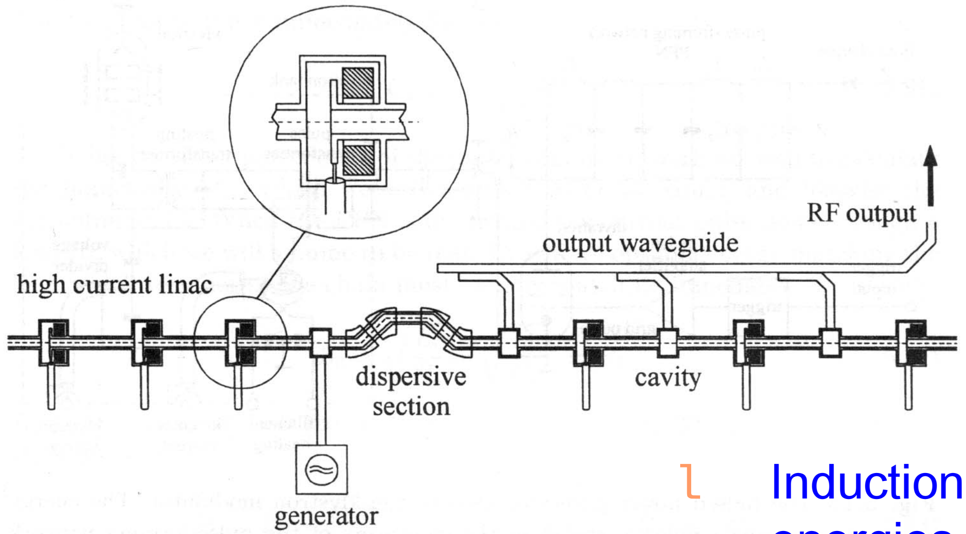
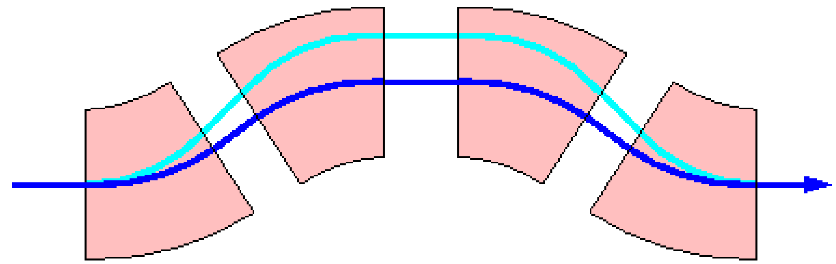
Relativistic Klystron

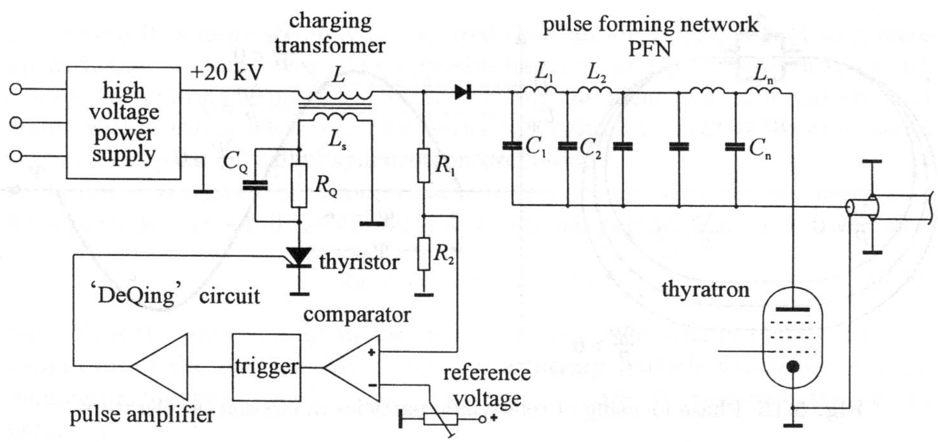




Bunch compressor with dispersion.

- E a few MeV
- I a few 1000A

- Induction linac for high currents and low energies.
- A high current low energy beam creates the RF power for a low current high energy beam.



A modulator pulses the Klystron in with the required repetition rate.

DeQing circuit precisely defines the voltage.

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Interaction Rate

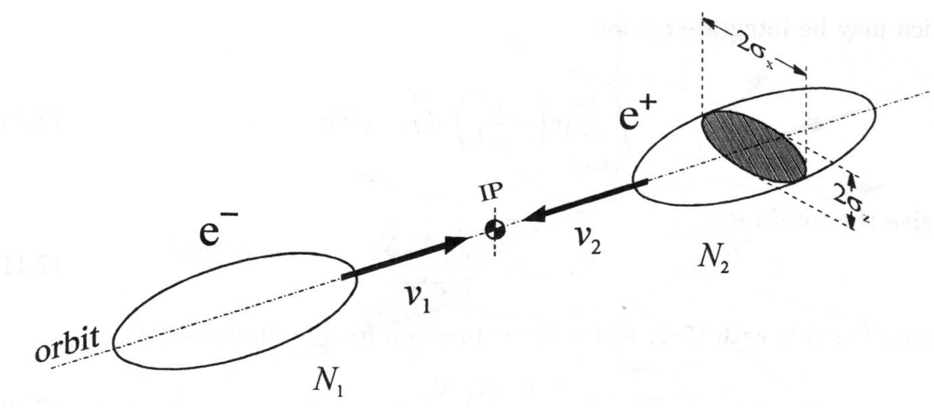
When the cross section for a process is known, the number of events per time is

$\dot{N}_{\text{events}} = L \cdot \sigma_{\text{cross section}}$ where the luminosity L is independent of the process.

$$L[\frac{1}{\text{cm}^2\text{s}}] = L\,10^{33}[\frac{1}{nb\ s}]$$

Integrated Luminosity: $\int L\,dt = N_{\text{events}} / \sigma_{\text{cross section}}$

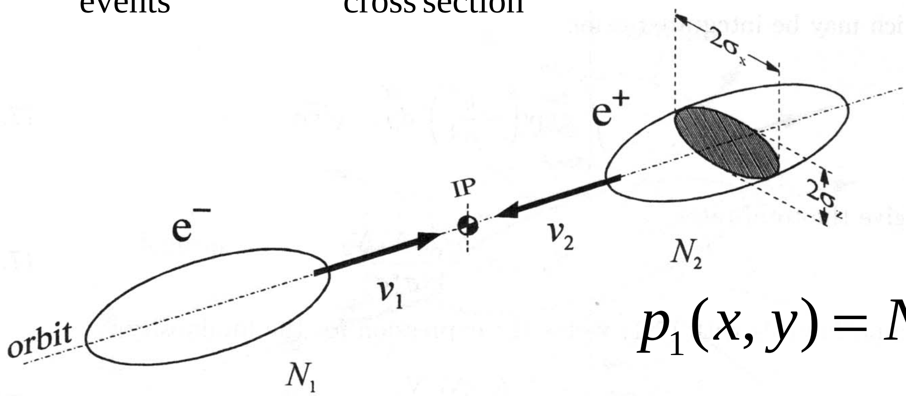
Gaussian beams:



Luminosity

When the cross section for a process is known, the number of events per time is

$\dot{N}_{\text{events}} = L \cdot \sigma_{\text{cross section}}$ where the luminosity L is independent of the process.



$$p_1(x,y) = N_2 \int \rho_2(x,y,\tau)\, d\tau \cdot \sigma_{\text{cross section}}$$

$$N_{events} = \sum p_1 = N_1 \int \rho_1(x,y,\tau)\, p_1(x,y) dx\, dy\, d\tau$$

$$L = \frac{N_1 N_2}{\Delta t} \int [\int \rho_1(x,y,\tau)\, d\tau \int \rho_2(x,y,\tau)\, d\tau]\, dx\, dy$$

Gaussian beams:

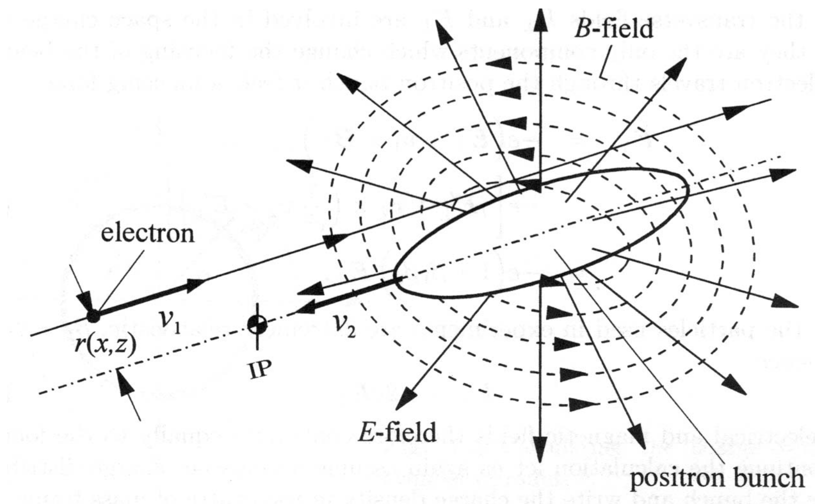
$$L = \frac{N_{\text{bunch}} f_0 N_1 N_2}{4\pi^2 \sigma_{1y}\sigma_{1x}\sigma_{2y}\sigma_{2x}} \int e^{-\frac{x^2}{2\sigma_{1x}^2}} e^{-\frac{x^2}{2\sigma_{2x}^2}} e^{-\frac{y^2}{2\sigma_{1y}^2}} e^{-\frac{y^2}{2\sigma_{2y}^2}} dxdy$$

$$= \frac{N_{\text{bunch}} f_0 N_1 N_2}{2\pi \sqrt{\sigma_{1x}^2 + \sigma_{2x}^2}\sqrt{\sigma_{1y}^2 + \sigma_{2y}^2}} = \frac{N_{\text{bunch}} f_0 N_1 N_2}{2\pi \Sigma_x \Sigma_y} = \frac{1}{N_{\text{bunch}} f_0} \frac{I_1 I_2}{2\pi e^2 \Sigma_x \Sigma_y}$$

The Beam-Beam Force

The force that acts from one beam to the other during collisions is focusing or defocusing in both planes for small distances.

For large distances it is very nonlinear and contributes to the dynamic aperture.



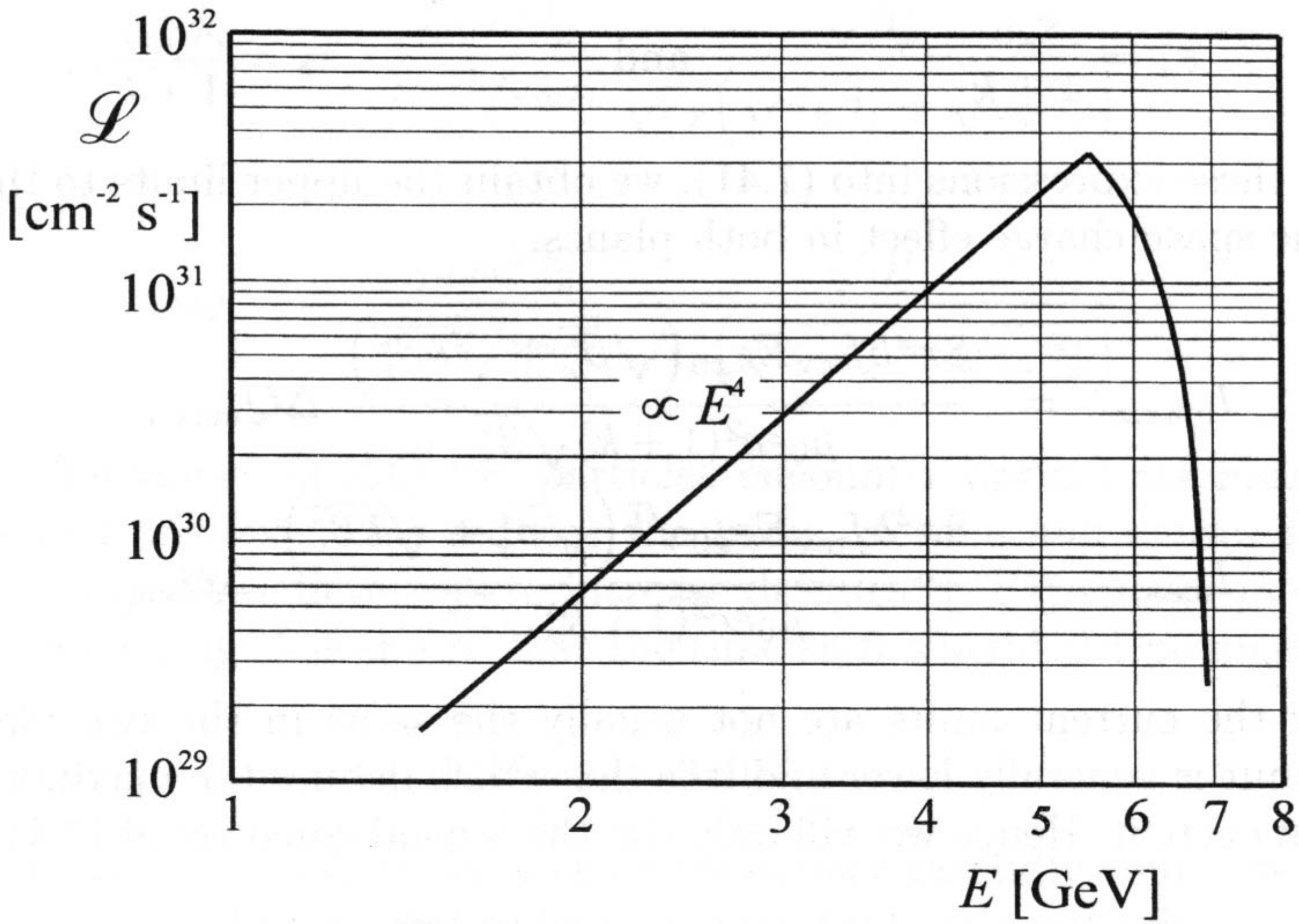
The effects of E and B forces add. Whereas they subtract for co-moving particles.

$$\Delta \upsilon_x^{(1)} = \frac{e^2 N^{(2)}}{8\pi^2 p^{(1)} c \varepsilon_0} \frac{\beta_x^{(1)}}{\sigma_x^{(2)}(\sigma_x^{(2)} + \sigma_y^{(2)})}$$

One should operate so that $\Delta \upsilon_x^{(1)} \leq 0.04$

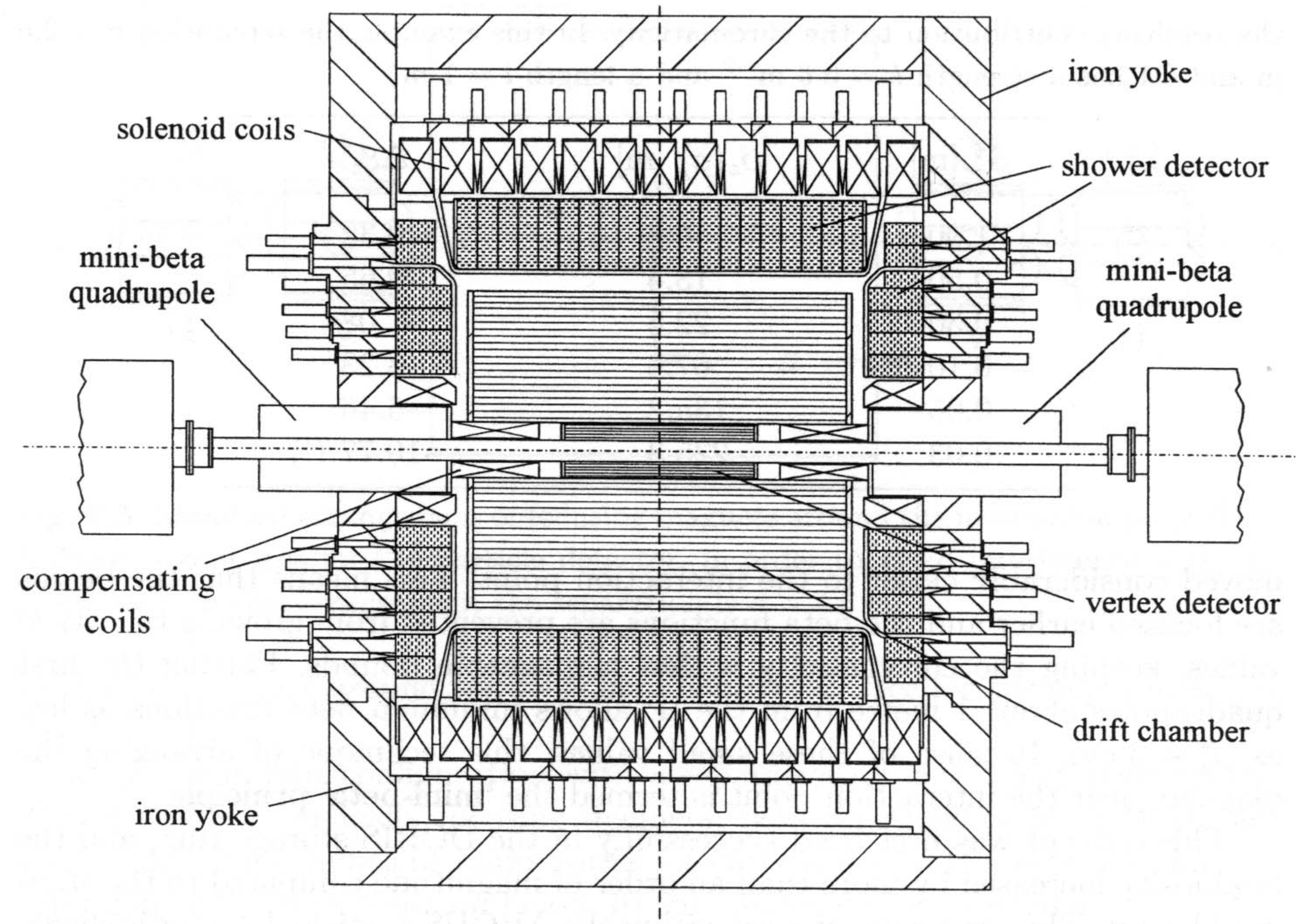
Limits to the Luminosity

If one stays at the beam beam tune shift limit, the luminosity grows with the 4th power of E, until for example the RF becomes too weak.



The Detector and the IR

The interaction region IR connects the accelerator intimately with the detector.



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